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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This report considers the assessment of von Neumann-Morgenstern utility functions u defined on two attributes from the viewpoint of mathematical approximation theory. It focuses on approximations v of u that can be written as $v(x,y) = f_{(1)}(x)g_{(1)}(y) + \dots + f_{(m)}(x)g_{(m)}(y)$ where the $f_{(i)}$ and/or $g_{(i)}$ are based on single-attribute conditional utility functions. The paper does not rely on independence axioms that enable $u(x,y)$ to be expressed in simplified forms when they are valid.					

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TECHNICAL REPORT NO. 20

APPROXIMATIONS OF TWO-ATTRIBUTE UTILITY FUNCTIONS

Peter C. Fishburn
 College of Business Administration
 The Pennsylvania State University
 University Park, Pennsylvania 16802

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1. INTRODUCTION

Let \succ ("is preferred to") be a binary relation on the set \mathcal{P} of simple probability measures or 'gambles' defined on a set T of consequences.

Throughout this study it will be assumed that:

1. T is the Cartesian product of two or more nondegenerate closed and bounded real intervals;

2. \succ on \mathcal{P} satisfies the axioms of von Neumann and Morgenstern (1947) or an equivalent system (Herstein and Milnor, 1953; Fishburn, 1970) so that there exists $u: T \rightarrow \mathbb{R}$ such that

$$P \succ Q \text{ iff } \sum_T P(t)u(t) > \sum_T Q(t)u(t), \text{ for all } P, Q \in \mathcal{P}, \quad (1)$$

with u^* satisfying (1) when u does iff u^* is a positive affine transformation of u of the form $u^* = u^{ab}$ where $u^{ab}(t) = au(t) + b$, $a > 0$;

3. The von Neumann-Morgenstern utility function u in (1) is continuous (Grandmont, 1972; Foldes, 1972) in the relative usual product topology for T .

The purpose of the study is to analyze methods for approximating u . The present paper deals with two-dimensional consequence spaces $T = X \times Y$; a sequel paper will examine $T = T_1 \times T_2 \times \dots \times T_n$ with $n \geq 3$. Although continuity of u is not implied by the axioms that are necessary and sufficient for (1), and plausible examples of discontinuities are easy to imagine, continuity is crucial to most work in approximation theory (e.g. Cheney, 1966, and Lorentz, 1966) and will be assumed here.

It is one thing to postulate the existence of continuous u on T that satisfies (1) and is unique up to transformations u^{ab} with $a > 0$, but quite another thing to estimate u in an actual decision situation. Consequently, much effort has been devoted to theoretical and methodological aspects of utility function assessment. The assessment and approximation of single-variable utility functions is discussed by Meyer and Pratt (1968), Bradley and Frey (1975), and Ohlson and Kallio (1975) among others. Theoretical work in multiattribute or multivariate utility (Fishburn, 1965, 1974; Pollak, 1967; Raiffa, 1969; Keeney, 1971, 1972a; Fishburn and Keeney, 1974; Farquhar, 1975) has focused on axioms for \succ on \mathcal{P} that allow $u(t_1, \dots, t_n)$ to be written as a combination of functions defined on fewer than n attributes, such as $u(t_1, t_2, \dots, t_n) = u_1(t_1) + u_2(t_2) + \dots + u_n(t_n)$ or $u(t_1, t_2, \dots, t_n) = u_1(t_1)u_2(t_2)\dots u_n(t_n)$. A desire to simplify the task of utility assessment has motivated much of this work. Examples of its application to specific situations are given by Raiffa (1969), Keeney (1972b, 1973) and Keeney and Nair (1974). Sicherman (1975) has developed an interactive computer program for assessment in the additive and multiplicative cases.

Research workers who have been involved in the development of special forms for multiattribute utility functions realize of course that the independence axioms that characterize the special forms may fail to hold in a given situation because of evaluative interdependencies among the attributes. Consequently, there is a need to explore the general problem of multiattribute utility assessment in the absence of simplifying independence assumptions. Although there are several approaches to this problem, the present study will focus on approximations of u that are written as finite sums of products of functions on the individual attributes. In the context of $T = X \times Y$, the form of approximation that will be used here is

$$v(x,y) = \sum_{i=1}^m f_i(x)g_i(y). \quad (2)$$

The only direct assessment of u that will be required for (2) involves either the evaluation of u at a finite number of points in T or the evaluation of a finite number of single-variable conditional utility functions of the form $u(x, y_j)$ and $u(x_i, y)$, where x_i is a fixed element in X and y_j is a fixed element in Y . The functions f_i and g_i may involve the conditional utility functions or they may be specified independently of any utility assessment. In later sections it will be assumed--as a first approximation--that there is no error in the assessment of the u values used in the right hand side of (2).

There are three main reasons for using (2). First, it is generally conceded that it is much easier to assess single-attribute utility functions than to assess two-attribute functions in their full generality. Secondly, the sum-of-products form of (2) is computationally attractive in the context of optimization algorithms. Finally, the right hand side of (2) subsumes the special forms of $u(x,y)$ that have thus far been characterized by independence axioms (Fishburn, 1974). The most general of these is $u(x,y) = u_1(x) + u_2(y) + h_1(x)h_2(y)$, which agrees with (2) when $m = 3$ and $f_1 = u_1$, $f_2 \equiv 1$, $f_3 = h_1$, $g_1 \equiv 1$, $g_2 = u_2$ and $g_3 = h_2$.

The present paper is organized as follows. Some basic ideas from approximation theory are briefly noted in the next section, and a result from this theory is presented in our utility context. The simple additive and multiplicative approximations are examined in section 3. Section 4 then discusses some elementary interpolation approximations based on finite sets of single-variable conditional utility functions. The final section examines

approximations that are exact on a grid in $X \times Y$: that is, they give $v(x_i, y) = u(x_i, y)$ for every x_i in a finite subset of X and all $y \in Y$, along with $v(x, y_j) = u(x, y_j)$ for every y_j in a finite subset of Y and all $x \in X$.

Readers who are familiar with the diversity and scope of approximation theory will realize that the present study represents a very modest step in the development of a theory and methodology for the approximation and assessment of multiattribute utility functions. It is hoped that the study will elicit additional interest in the topic.

2. REMARKS ON APPROXIMATION THEORY

This section outlines a few basic ideas of approximation theory, comments on aspects of (2) that will play a role throughout the paper, and provides an application of approximation theory to our utility context. A broader introduction to approximation theory can be obtained from the paper by Buck (1959) and the books by Lorentz (1966) and Cheney (1966). Other suggested works include the papers by Rivlin and Shapiro (1961), Lorentz (1972) and Jerome (1973), the collections edited by Langer (1959) and Lorentz (1973), and various articles in the Journal of Approximation Theory.

Appropriate to our purposes let $S = [0, 1]^n$ be the n -dimensional unit cube and let $C(S)$ be the real linear space (Kelley and Namioka, 1963) of all continuous real valued functions on S . The most commonly used norm in approximation theory for measuring distances between functions in $C(S)$ is the uniform norm

$$\|f\| = \sup_{s \in S} |f(s)| = \max_S |f(s)|, \quad f \in C(S), \quad (3)$$

approximations that are exact on a grid in $X \times Y$: that is, they give $v(x_i, y) = u(x_i, y)$ for every x_i in a finite subset of X and all $y \in Y$, along with $v(x, y_j) = u(x, y_j)$ for every y_j in a finite subset of Y and all $x \in X$.

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$$\|f\| = \sup_{s \in S} |f(s)| = \max_S |f(s)|, \quad f \in C(S), \quad (3)$$

the latter equality arising from compactness and continuity. (Least squares minimization problems use the quadratic or Euclidean norm.) The uniform norm will be used throughout our study. Convergence in this norm, i.e. $\|f_m - f\| \rightarrow 0$ as $m \rightarrow \infty$ with $f, f_1, f_2, \dots \in C(S)$, is equivalent to the uniform convergence of the f_m to f .

Let $f \in C(S)$ be given and let D be a nonempty subset of $C(S)$ whose functions are proposed as approximations of f . Define $d(D, f) = \inf_{g \in D} \|g - f\|$ and let $D_f = \{g \in D: \|g - f\| = d(D, f)\}$. Hence $d(D, f)$ is the 'distance' from D to f and the functions in D_f provide the best approximations to f as gauged by (3). Among other things, approximation theory is concerned with the following questions:

1. Can $d(D, f)$ be specified precisely, or is it possible to obtain tight bounds on it?
2. What can be said about the functions in D for which $\|g - f\| < d(D, f) + \delta$ for given $\delta > 0$?
3. Is D_f nonempty, and, if so, can its structure and/or specific contents be identified?

The set D is frequently taken to consist of a finite-dimensional subspace of $C(S)$ generated by a basis of linearly independent functions g_1, \dots, g_N , in which case $D = \left\{ \sum_{k=1}^N a_k g_k: a_k \in \text{Re for all } k \right\}$. For example, D might consist of all polynomials in the n variables of highest degree r , or, in our utility context, D might be based on conditional univariate utility functions. When D is a finite-dimensional linear subspace of $C(S)$, D_f is convex and nonempty (e.g., Buck, 1959, Theorem 2).

Buck (1959) illustrates the above ideas with a specific example using $n = 2$ and $N = 5$. Let $f(x, y) = xy$ and let g_1 through g_5 in $C(S)$ be

respectively the identity function ($g_1 \equiv 1$), x , y , x^2 and y^2 . Buck shows that $d(D, f) = 1/4$ and that $D_f = \{\alpha f_1 + (1 - \alpha)f_2 : 0 \leq \alpha \leq 1\}$ where $f_1(x, y) = (x^2 + y^2)/2 - 1/4$ and $f_2(x, y) = x + y - (x^2 + y^2)/2 - 1/4$.

When D is an infinite-dimensional but 'small' subspace of $C(S)$, it may be true that $d(D, f) = 0$ for every $f \in C(S)$ although D_f must then be empty for 'most' $f \in C(S)$. A useful theorem developed by Bohman and Korovkin (Korovkin, 1959; Lorentz, 1966, p. 7) shows that $d(D, f) = 0$ for all $f \in C(S)$ can sometimes be established by showing only that $d(D, f) = 0$ for a finite number of specific functions f . Consider, for example, the famous Weierstrass approximation theorem (Lorentz, 1966, p. 10), which says that $d(D, f) = 0$ for all $f \in C(S)$ when $S = \{(t_1, \dots, t_n) : 0 \leq t_i \leq 1\}$ and D is the space of all ordinary algebraic polynomials in t_1, \dots, t_n . Using the Bernstein polynomials

$$B_m(f; t_1, \dots, t_n) = \sum_{k_1=0}^m \dots \sum_{k_n=0}^m f\left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right) b_{mk_1}(t_1) \dots b_{mk_n}(t_n), \quad (4)$$

where $b_{mk}(t_i) = \binom{m}{k} t_i^k (1 - t_i)^{m-k}$, the Bohman-Korovkin theorem allows one to prove the Weierstrass theorem by showing that $B_m(f) \rightarrow f$ as $m \rightarrow \infty$ for each of the $2n + 1$ functions $1, t_i, t_i^2, (i = 1, \dots, n)$.

Although the above discussion barely scratches the surface of approximation theory, it will suffice for our present purposes. We now return to our concern with two-attribute utility functions.

Considerations in Utility Approximations

Throughout the remainder of this paper we shall take $T = X \times Y = [0, 1]^2$ without loss of generality since the closed and bounded intervals for the two attributes can be mapped linearly onto $[0, 1]$. Two different types of

approximations of continuous u on T by (2) will be considered according to whether the approximation attempts to provide exact values of u at the points in T at which u is assessed directly. (See the discussion following (2).) Cases in which $v(x,y) = u(x,y)$ for certain designated points in T or for certain conditional utility functions on X or Y are cases of approximation by interpolation. Some elementary interpolation methods are presented in the ensuing sections.

The proof method mentioned above for the Weierstrass theorem is not based on interpolation since the Bernstein polynomials, which in the present context can be written as

$$v_m(x,y) = \sum_{i=0}^m \sum_{j=0}^m u(i/m, j/m) \binom{m}{i} x^i (1-x)^{m-i} \binom{m}{j} y^j (1-y)^{m-j}, \quad (5)$$

do not generally yield $v_m(i/m, j/m) = u(i/m, j/m)$. In fact, as shown by Faber (1914), the Weierstrass theorem cannot be proved using interpolating polynomials. Since the Weierstrass theorem shows that $\|v_m - u\| \rightarrow 0$ as $m \rightarrow \infty$, (5) provides a specific instance for (2) in which the approximation for u becomes precise in the limit. Nevertheless, it appears (Lorentz, 1966, p. 102) that the convergence of v_m to u is rather slow compared to the convergence obtained by polynomials in $D_u^{(m)}$, where $D^{(m)} = \{ \sum_{i=0}^m \sum_{j=0}^m a_{ij} x^i y^j : a_{ij} \in \text{Re} \}$.

Approximations (2) can also be classified according to whether v undergoes the same affine transformation as does u when u on the right hand side of (2) is replaced by $u^{ab} = au + b$, $a > 0$. Given $v(x,y) = \sum f_i(x)g_i(y)$ as in (2), we shall let v^{ab} denote the function obtained from $\sum f_i(x)g_i(y)$ when every instance of u in this expression is replaced by u^{ab} . It is

important to realize that v^{ab} need not be equal to $av + b$. We shall say that v is affine preserving at (a,b) if and only if $v^{ab}(x,y) = av(x,y) + b$ for all $(x,y) \in T$, and that v is affine preserving if and only if it is affine preserving at (a,b) for all $a > 0$ and all real b . Since

$$\begin{aligned} v_m^{ab}(x,y) &= \sum_i \sum_j [au(i/m, j/m) + b] \binom{m}{i} x^i (1-x)^{m-i} \binom{m}{j} y^j (1-y)^{m-j} \\ &= av_m(x,y) + b, \end{aligned}$$

the approximation (5) is affine preserving. The approximations considered in ensuing sections are affine preserving when $b = 0$, but several are not generally affine preserving at (a,b) when $b \neq 0$.

We shall also be interested in whether v is monotonic in x and/or y when u is monotonic in x and/or y . If v is monotonic increasing in x whenever u is monotonic increasing in x , and if v is monotonic decreasing in x whenever u is monotonic decreasing in x , then we shall say that v is monotonicity preserving in x . A similar convention holds for y . In addition, v is said to be monotonicity preserving if it is monotonicity preserving in each variable.

3. ELEMENTARY APPROXIMATIONS

This section examines simple additive, multiplicative, and additive-multiplicative forms for v . Conditions for \succ on \mathcal{P} under which these forms are exact (i.e. $\|v - u\| = 0$) when the functions involved in (2) are properly aligned are presented in Fishburn (1974) and will not be repeated here with the exception of a comment following the proof of Theorem 5.

The Simple Additive Form

We begin with the simple additive form. Recall that $T = X \times Y = [0,1]^2$.

THEOREM 1. Given fixed $(x_1, y_1) \in [0,1]^2$ suppose that

$$v(x,y) = u(x,y_1) + u(x_1,y) - u(x_1,y_1), \text{ for all } (x,y) \in T. \quad (6)$$

Then $v(x,y) = u(x,y)$ if $x = x_1$ or $y = y_1$, and v is affine preserving and monotonicity preserving. In addition, with $W = \max_T u(x,y) - \min_T u(x,y)$, (x_1, y_1) can be chosen so that

- (a) $||v - u|| \leq (3/2)W$,
- (b) $||v - u|| \leq W$ if u is monotonic in either x or y ,
- (c) $||v - u|| \leq (1/2)W$ if u is monotonic in both variables.

If u is constant then $W = 0$ and (a) through (c) hold with equality. In general $||v - u|| = 0$ iff $u(x,y) + u(x_1, y_1) = u(x,y_1) + u(x_1, y)$ for all $(x,y) \in T$, regardless of how (x_1, y_1) is chosen. It may also be noted that the bound in (c) is satisfied when the constant approximation $v(x,y) = (\max u + \min u)/2$ is used, regardless of whether u is monotonic. However, this approximation does not have the form of (6).

Proof. The assertions in the first part of Theorem 1 are obvious from (6). The heavy line segments in Figure 1(a) show where v must equal u . To

Figure 1 about here

prove the latter part of the theorem, observe that

$$|v(x,y) - u(x,y)| = |u(x,y) + u(x_1, y_1) - u(x, y_1) - u(x_1, y)|,$$

$$||v - u|| = \max_T |v(x,y) - u(x,y)|$$

For notational convenience suppose that u is not constant and let $\max_T u(x,y) = 1$ and $\min_T u(x,y) = 0$. Choose (x_1, y_1) so that $u(x_1, y_1) = 1/2$. Then $||v - u|| \leq 3/2$. If u is monotonic in x then, when (x,y) is in region I of Figure 1(a),

$$|v(x,y) - u(x,y)| \leq \max \{|u(x,y) - u(x_1, y_1)|, |u(x_1, y_1) - u(x, y_1)|\} \leq 1,$$

and similar calculations in the other three regions show that $||v - u|| \leq 1$. The same conclusion holds if u is monotonic in y . Finally, suppose that u is monotonic. We consider explicitly only the case in which u increases in x and decreases in y : other cases are left to the reader. By examining signs of u differences it follows that, when (x,y) is in region I of Figure 1(a),

$$\begin{aligned} |v(x,y) - u(x,y)| &\leq \min \{ \max \{ u(x_1, y) - u(x,y), u(x, y_1) - u(x, y_1) \}, \\ &\quad \max \{ u(x,y) - u(x, y_1), u(x_1, y) - u(x_1, y_1) \} \} \\ &\leq \max \{ 1/2, \min \{ u(x_1, y) - u(x,y), u(x,y) - u(x, y_1) \} \} = 1/2 \end{aligned}$$

since $\min \{ u(x_1, y) - u(x,y), u(x,y) - u(x, y_1) \} \leq 1/2$. When (x,y) is in region II of Figure 1(a), $|v(x,y) - u(x,y)| \leq \min \{ \max \{ u(x,y) - u(x, y_1), u(x, y_1) - u(x_1, y_1) \}, \max \{ u(x,y) - u(x, y_1), u(x_1, y) - u(x_1, y_1) \} \} \leq \max \{ 1/2, \min \{ u(x,y) - u(x_1, y), u(x,y) - u(x, y_1) \} \} = 1/2$ since $1/2 \leq \min \{ u(x, y), u(x, y_1) \} \leq \max \{ u(x_1, y), u(x, y_1) \} \leq u(x,y) \leq 1$. The region III and IV analyses are similar respectively to those for regions I and II. The affine transformation u^{wb} on u for which $\max u = 1$ and $\min u = 0$ then gives conclusions (a), (b) and (c) as stated in the theorem. Q.E.D.

The preceding proof chose (x_1, y_1) so that $u(x_1, y_1)$ is midway between $\min u$ and $\max u$. Despite the fact that many points in T have a u value that

is midway between the extremes of u , it does not follow that one of these will minimize $||v - u||$ when v is given by (6). Although it is not generally possible to select (x_1, y_1) to minimize $||v - u||$ without knowing u on all of T , there are cases in which this might be done when u is not completely known. To illustrate this, we shall say that u is conservative when it is strictly increasing in both variables and

$$\{0 \leq x < x^* \leq 1, 0 \leq y < y^* \leq 1\} \Rightarrow u(x, y^*) + u(x^*, y) > u(x, y) + u(x^*, y^*). \quad (7)$$

If approximation v is conservative whenever u is conservative, then v will be said to be conservatism preserving. Since (6) gives $v(x, y^*) + v(x^*, y) = v(x, y) + v(x^*, y^*)$, it is not conservatism preserving.

A plausible example of conservatism (Fishburn, 1973; Richards, 1973) arises when the pairs in T are two-period income streams. Then (7) holds if the even-chance gamble between (x, y^*) and (x^*, y) , which ensures one of the larger amounts x^* or y^* , is preferred to the even-chance gamble between (x, y) and (x^*, y^*) , which could (pr. = 1/2) result in the lower amounts x and y in both periods.

THEOREM 2. Suppose that u is conservative and v is given by (6). Let $||v - u||_{(x_1, y_1)}$ denote the value of $||v - u||$ when (x_1, y_1) is used as the fixed point in (6) and let

$$\Delta = u(1, 0) + u(0, 1) - u(0, 0) - u(1, 1).$$

Then (x_1, y_1) can be chosen so that $\Delta/4 \leq ||v - u||_{(x_1, y_1)} < \Delta/3$, and it is impossible to have $||v - u|| < \Delta/4$. Also let

$$\psi_{ij}(x, y) = (-1)^{i+j} [u(x, j) + u(i, y) - u(x, y) - u(i, j)] \quad (8)$$

for $i \in \{0,1\}$ and $j \in \{0,1\}$. Then $\varphi_{ij} \geq 0$ for each of the four (i,j) pairs,
and

$$||v - u||_{(x,y)} = \max \{ \varphi_{00}(x,y), \varphi_{11}(x,y), \varphi_{01}(x,y), \varphi_{10}(x,y) \}. \quad (9)$$

Moreover, there exists a unique $(x^*, y^*) \in T$ that satisfies $\varphi_{00}(x,y) =$
 $\varphi_{11}(x,y)$ and $\varphi_{01}(x,y) = \varphi_{10}(x,y)$, and the unique x^* and y^* are specified by

$$u(x^*, 1) - u(x^*, 0) = \frac{1}{2} [u(1, 1) - u(1, 0)] + \frac{1}{2} [u(0, 1) - u(0, 0)] \quad (10)$$

$$u(1, y^*) - u(0, y^*) = \frac{1}{2} [u(1, 1) - u(0, 1)] + \frac{1}{2} [u(1, 0) - u(0, 0)]. \quad (11)$$

Given (x^*, y^*) as specified:

(i) if $\varphi_{00}(x^*, y^*) = \varphi_{01}(x^*, y^*)$ then $||v - u||_{(x^*, y^*)} = \Delta/4$;

(ii) if $\varphi_{00}(x^*, y^*) < \varphi_{01}(x^*, y^*)$ then $||v - u||$ is minimized by a
point that satisfies $\varphi_{00}(x,y) = \varphi_{10}(x,y)$; and $\varphi_{01}(x,y) = \varphi_{10}(x,y) = \varphi_{00}(x,y)$
and $\varphi_{01}(x,y) = \varphi_{10}(x,y) = \varphi_{11}(x,y)$ are respectively satisfied by points
interior to T at which $||v - u|| < \Delta/3$;

(iii) if $\varphi_{00}(x^*, y^*) > \varphi_{01}(x^*, y^*)$ then $||v - u||$ is minimized by a
point that satisfies $\varphi_{00}(x,y) = \varphi_{11}(x,y)$; and $\varphi_{00}(x,y) = \varphi_{11}(x,y) = \varphi_{01}(x,y)$
and $\varphi_{00}(x,y) = \varphi_{11}(x,y) = \varphi_{10}(x,y)$ are respectively satisfied by points
interior to T at which $||v - u|| < \Delta/3$.

The part of Theorem 2 that precedes (8) simply summarizes assertions spelled out in greater detail following (8). Using (7), the four equations of (8) give the values of $|v - u|$ at the four corners of T when (x, y) is the fixed point in (6). Equation (9) says that the largest value of $|v - u|$ occurs at one of the four corners of T for every choice of fixed point in (6). Since as is easily checked,

$$\varphi_{00}(x,y) + \varphi_{11}(x,y) + \varphi_{01}(x,y) + \varphi_{10}(x,y) = \Delta,$$

it is impossible to have $||v - u|| < \Delta/4$. The final part of the theorem shows how a fixed point for (6) can be identified so that $||v - u|| < \Delta/3$ regardless of the nature of u so long as it is conservative. The interior point qualification is used in (ii) and (iii) since, for example, $\varphi_{01}(0,0) = \varphi_{10}(0,0) = \varphi_{00}(0,0) = 0$, in which case $||v - u||_{(0,0)} = \varphi_{11}(0,0) = \Delta$.

Proof. Using (7) it is easily seen that $\varphi_{ij} \geq 0$ for all $i, j \in \{0,1\}$ and that $\varphi_{ij}(x,y) > 0$ whenever (x,y) is on the interior of T . To verify (9), suppose first that (x_1, y_1) is the fixed point for (6) and that (x_2, y_2) lies in region III of Figure 1(a), with $x_1 < x_2 \leq 1$ and $y_1 < y_2 \leq 1$, and $(x_2, y_2) \neq (1,1)$. Then, by conservatism,

$$\begin{aligned} u(x_2, 1) + u(1, y_2) &\geq u(x_2, y_2) + u(1, 1) \\ u(x_1, 1) + u(x_2, y_2) &\geq u(x_1, y_2) + u(x_2, 1) \\ u(x_2, y_2) + u(1, y_1) &\geq u(x_2, y_1) + u(1, y_2) \end{aligned}$$

with at least one strict inequality. Addition of these three inequalities then gives $u(x_1, 1) + u(1, y_1) - u(x_1, y_1) - u(1, 1) > u(x_2, y_2) + u(x_1, y_2) - u(x_1, y_1) - u(x_2, y_2)$, or $\varphi_{11}(x_1, y_1) > \varphi_{11}(x_2, y_2) - u(x_2, y_2)$. Consequently $|v(x,y) - u(x,y)|$ is maximized in region III at $(x,y) = (1,1)$. Similar analyses in each of the other three regions of Figure 1(a) shows that, regardless of the choice of (x_1, y_1) for (6), $|v - u|$ is maximized at one of the four corners of T , and (9) then follows immediately.

To verify (10) and (11) and prepare for the final assertions in the theorem, we write $\varphi_{01}(x,y) = \varphi_{10}(x,y)$ and $\varphi_{01}(x,y) = \varphi_{10}(x,y)$ in terms of u , using (8), to obtain respectively

$$u(1,y) - u(0,y) = [u(1,1) - u(0,0)] - [u(x,1) - u(x,0)], \quad (12)$$

$$u(1,y) - u(0,y) = [u(1,0) - u(0,1)] + [u(x,1) - u(x,0)]. \quad (13)$$

Conservatism implies that $u(1,y) - u(0,y)$ decreases in y and that $u(1,x) - u(0,x)$ decreases in x . Thus, for each x , (12) will be satisfied by a unique y that decreases as x increases, with $y = 1$ when $x = 0$ and $y = 0$ when $x = 1$. Similarly, (13) has $y = 0$ when $x = 0$ and $y = 1$ when $x = 1$ with the unique y solution for each x increasing as x increases. The resulting curves in T described by (12) and (13) are shown in Figure 2. Their unique point of

Figure 2 about here

intersection is (x^*, y^*) as specified in (10) and (11). This point is given by the joint solution of (12) and (13). If $\varphi_{00}(x^*, y^*) = \varphi_{01}(x^*, y^*)$ then all four φ_{ij} are equal to $\Delta/4$ at (x^*, y^*) and this point uniquely minimizes $\|v - u\|$, as specified in (i) of the theorem, following (11).

The following lemma, which is a variation on the theme of the first part of this proof, will be used in dealing with (ii) and (iii).

LEMMA 1. If $(i,j) \neq (x_1, y_1) \neq (x_2, y_2)$ and (x_2, y_2) is in the rectangle (or straight line segment if $i = x_1$ or $j = y_1$) two corners of which are (x_1, y_1) and (i,j) , then $\varphi_{ij}(x_1, y_1) > \varphi_{ij}(x_2, y_2)$.

Proof. Suppose first that $(i,j) = (1,1)$ with (x_1, y_1) and (x_2, y_2) as pictured in Figure 1(a). Then, by conservatism,

$$u(x_1, 1) + u(x_2, y_2) \geq u(x_1, y_2) + u(x_2, 1)$$

$$u(x_1, y_2) + u(x_2, y_1) \geq u(x_1, y_1) + u(x_2, y_2)$$

$$u(1, y_1) + u(x_2, y_2) \geq u(x_2, y_1) + u(1, y_2)$$

with at least one strict inequality. Addition of these inequalities gives $u(x_1, 1) + u(1, y_1) - u(x_1, y_1) - u(1, 1) > u(x_2, 1) + u(1, y_2) - u(x_2, y_2) - u(1, 1)$, or $\varphi_{11}(x_1, y_1) > \varphi_{11}(x_2, y_2)$. The other three regions in Figure 1(a) are handled in a similar fashion. Q.E.D.

We now return to (ii) and (iii) of the theorem. Consider the point labeled Q_0 in Figure 2. By Lemma 1, $\varphi_{00}(Q_0) > \varphi_{01}(Q_0)$ and $\varphi_{11}(Q_0) > \varphi_{10}(Q_0)$, and, since $\varphi_{00}(Q_1) = \varphi_{11}(Q_1)$, $\varphi_{00}(Q_0) > \varphi_{11}(Q_0)$. Similarly, using Q_0 and Q_2 , $\varphi_{10}(Q_0) > \varphi_{01}(Q_0)$. By Lemma 1, $\varphi_{00}(Q_0)$ and $\varphi_{10}(Q_0)$ will be reduced when the fixed point for (6) is moved from Q_0 in the direction of the arrow emanating from Q_0 , and therefore $\|v - u\|$ cannot be minimized by taking Q_0 as the fixed point for (6). Similar results apply in the other three regions of Figure 2. Consequently, $\min_T \|v - u\|_{(x,y)}$ must occur at a point on one of the $\varphi_{00} = \varphi_{11}$ and $\varphi_{01} = \varphi_{10}$ curves. We have already noted that the (unique) minimizing point will be (x^*, y^*) if $\varphi_{00}(x^*, y^*) = \varphi_{01}(x^*, y^*)$.

Suppose as in (ii) that $\varphi_{00}(x^*, y^*) < \varphi_{01}(x^*, y^*)$. Then, by Lemma 1, $\|v - u\|_{(x^*, y^*)} < \|v - u\|_{(x,y)}$ for every $(x, y) \neq (x^*, y^*)$ on the $\varphi_{00} = \varphi_{11}$ curve. (A move from (x^*, y^*) towards $(0, 1)$ will increase φ_{00} , which was one of the maximizing φ_{ij} at (x^*, y^*) .) Therefore a fixed point for (6) that minimizes $\|v - u\|$ must lie on the $\varphi_{01} = \varphi_{10}$ curve. By hypothesis in this paragraph, $\varphi_{00}(x^*, y^*) = \varphi_{11}(x^*, y^*) < \varphi_{01}(x^*, y^*) = \varphi_{10}(x^*, y^*)$. As the fixed point for (6) moves upwards from (x^*, y^*) to $(1, 1)$ along the $\varphi_{01} = \varphi_{10}$ curve, $\varphi_{00}(x, y)$ increases continuously up to $\Delta = \varphi_{00}(1, 1)$, $\varphi_{11}(x, y)$ decreases to $0 = \varphi_{11}(1, 1)$, and $\varphi_{01}(x, y)$ and $\varphi_{10}(x, y)$, which are equal, may fluctuate but eventually arrive at zero at $(1, 1)$. It follows from continuity that there is a point on this part of the $\varphi_{01} = \varphi_{10}$ curve where $\varphi_{00} = \varphi_{11} = \varphi_{01} = \varphi_{10} > 1/4 > \varphi_{00} > 0$, e.g. Q_2 , and with this point as the fixed point for (6) we obtain

$\|v - u\| < \Delta/3$ with the use of (9). The remaining parts of the proofs of (ii) and (iii) are similar. Q.E.D.

If (7) is changed by reversing the inequality to $u(x,y) + u(x^*,y^*) > u(x,y^*) + u(x^*,y)$, which might suggest a 'daring' u instead of a 'conservative' u , then an obvious correspondent to Theorem 2 follows under appropriate sign changes.

Theorem 2 illustrates typical concerns of approximation theory as outlined after (3). For Theorem 2, D is the subset of $C(T)$ whose functions are given by (6) as the fixed point (x_1, y_1) ranges over T . Unlike most typical cases, D depends explicitly on u . Theorem 2 shows that $\Delta/4 \leq d(D, u) < \Delta/3$, and (9) implies that D_u is nonempty. Functions in or near to those in D_u were identified in the latter part of the theorem.

The Simple Multiplicative Form

The basic multiplicative approximation for u on $T = [0,1]^2$ can be expressed as $v(x,y) = f(x)g(y)$, as in (2). Although it is not necessary to align v with u in any specific way, we shall consider the case in which $v(x,y) = u(x,y)$ whenever $x = x_1$ or $y = y_1$, where (x_1, y_1) is a fixed point in T . This coincides with our alignment of the additive approximation (6). Given $v(x,y) = f(x)g(y)$ with u scaled so that $u(x_1, y_1) \neq 0$, the specified alignment implies that

$$v(x,y) = \frac{u(x,y)u(x_1,y_1)}{u(x_1,y_1)}, \quad \text{for all } (x,y) \in T. \quad (14)$$

Although this looks quite different than (6) and indeed is in most cases, we shall see momentarily that (6) is a limiting case of (14).

If u has constant sign then v as given by (14) is monotonic in x or y when u is monotonic in x or y , but if 0 is in the interior of the image of u then monotonicity preservation does not generally hold. Another difference between (14) and (6) is that (14) is not generally affine preserving. In particular, when (14) is used and $au(x_1, y_1) + b \neq 0$,

$$\begin{aligned} v^{ab}(x, y) &= \frac{[au(x, y) + b][au(x_1, y_1) + b]}{au(x_1, y_1) + b} \\ &= av(x, y) + b - \frac{ab[u_{11} - u(x, y)][u_{11} - u(x_1, y)]}{u_{11}(au_{11} + b)}, \end{aligned} \quad (15)$$

where for convenience we define $u_{11} = u(x_1, y_1)$. This shows that (14) is affine preserving at (a, b) if $b = 0$, but it is affine preserving when $b \neq 0$ only under very special conditions, i.e. when $u_{11} = u(x, y)$ or $u_{11} = u(x_1, y)$ for all (x, y) .

The essential nature of v^{ab} remains unchanged if it (rather than u) undergoes a positive affine transformation. In particular, if we subtract b from both sides of (15) and then divide by a , we obtain

$$\frac{v^{ab}(x, y) - b}{a} = v(x, y) + \left\{ \frac{-b}{u_{11}(au_{11} + b)} \right\} [u_{11} - u(x, y)][u_{11} - u(x_1, y)]. \quad (16)$$

For convenience, let $K = -b/[u_{11}(au_{11} + b)]$ and let $v_K(x, y)$ denote the left hand side of (16). Then (16) can be written as

$$v_K(x, y) = \frac{u(x, y)u(x_1, y)}{u_{11}} + K[u_{11} - u(x, y)][u_{11} - u(x_1, y)], \quad (17)$$

where K is any real number other than $-1/u_{11}$. The latter value of K is forbidden since it corresponds to $b = +\infty$ or $b = -\infty$ in (16). Any other real

value of K is obtainable from $K = -b/[u_{11}(au_{11} + b)]$, and infinite K is forbidden by the proscription against $au_{11} + b = 0$ in writing (15). It is easily checked that $v_K(x,y) = u(x,y_1) + u(x_1,y) - u(x_1,y_1)$, which is the additive form (6), when K in (17) is set equal to the forbidden value of $-1/u_{11}$. Hence, by choosing K for (17) arbitrarily near to $-1/u_{11}$, the multiplicative approximation (17) becomes arbitrarily close to the additive approximation (6); the convergence of (17) to (6) as K approaches $-1/u_{11}$ is easily seen to be uniform. Hence all results stated for the additive case apply, in the limit, to the multiplicative case.

It is important to note that the utility function u as used in (15), (16) and (17) is precisely the same function used in (14). Equation (17) simply describes the family of all basic multiplicative approximations--unique up to isomorphism under positive affine transformations on the approximations--that correspond to different ways of choosing an origin and scale unit for u and that render the approximation exact when $x = x_1$ or $y = y_1$. The parameter K in (17), unrestricted except by $K \neq -1/u_{11}$, describes the different approximations in this family. Naturally, if the fixed point (x_1, y_1) used in (14) or (17) is changed, then a different family of multiplicative approximations is described by (17).

The multiplicative approximation is more flexible than the additive approximation in the sense that, in addition to x_1 and y_1 , it has the parameter K that can be manipulated in fitting a simple multiplicative approximation to u . We shall investigate aspects of this flexibility in the next several theorems. The first of these follows immediately from the discussion following (17). Recall that $u_{11} = u(x_1, y_1)$.

THEOREM 3. Suppose u is naturally additive, so that $||v - u|| = 0$ when v is specified by (6), and $u_{11} \neq 0$. Then, for every $\delta > 0$, there is a $K \neq -1/u_{11}$ such that $||v_K - u|| < \delta$, where v_K is specified by (17).

In other words, any additive (and continuous) utility function u on T can be approximated arbitrarily closely by an appropriate multiplicative function. Note, however, that the converse of this is not true. Consider, for example, the naturally multiplicative function $u(x,y) = xy$. The general form of (6) for this case is $v(x,y) = cx + dy - cd$, with $c, d \in [0,1]$. As in Buck's example of the preceding section, the smallest value of $||v - u||$ obtainable in this case is $1/4$, which occurs when $c = d = 1/2$.

Another indication of the flexibility permitted by K is given by the following theorem, which discusses the possibility of making the multiplicative approximation exact at a point (x_2, y_2) for which $x_2 \neq x_1$ and $y_2 \neq y_1$, as pictured in Figure 1(a). For notational convenience we extend the previous convention of writing $u(x_1, y_1)$ as u_{11} by defining u_{ij} according to

$$u_{ij} = u(x_i, y_j). \quad (18)$$

THEOREM 4. Given u with $u_{11} \neq 0$, let v_K be defined by (17) and let (x_2, y_2) be a point in T at which $x_2 \neq x_1$ and $y_2 \neq y_1$. Then $v_K(x_2, y_2) = u(x_2, y_2) = u_{22}$ for some $K \neq -1/u_{11}$ if and only if either:

(a) $\{u_{11} = u_{21} \text{ and } u_{12} = u_{22}\}$ or $\{u_{11} = u_{12} \text{ and } u_{21} = u_{22}\}$, in which case $v_K(x_2, y_2) = u_{22}$ for every K ; or

(b) $u_{12} \neq u_{11} \neq u_{21}$ and $u_{11} + u_{22} \neq u_{12} + u_{21}$, in which case $v_K(x_2, y_2) = u_{22}$ if and only if $K = [u_{11} u_{22} - u_{12} u_{21}] / [u_{11}(u_{11} - u_{12})(u_{11} - u_{21})]$.

Proof. To satisfy $v_K(x_2, y_2) = u_{22}$ we require

$$u_{21} u_{12} + Ku_{11} [u_{11} - u_{21}] [u_{11} - u_{12}] = u_{11} u_{22}, \quad (19)$$

using (17). If $u_{11} = u_{21}$ then (19) holds if and only if $u_{12} = u_{22}$, and if $u_{11} = u_{12}$ then (19) holds if and only if $u_{21} = u_{22}$. In either case, $v_K(x_2, y_2) = u_{22}$ for every K . The only other possibility for u_{11} is $u_{12} \neq u_{11}$ and $u_{21} \neq u_{11}$, in which case (19) holds for the unique K specified in alternative (b) of the theorem. This value of K equals $-1/u_{11}$ if and only if $u_{11} [u_{11} u_{22} - u_{12} u_{21}] = -u_{11} (u_{11} - u_{12}) (u_{11} - u_{21})$, which reduces to $u_{11} + u_{12} = u_{12} + u_{21}$. Q.E.D.

Given (x_1, y_1) as the fixed point in (6) or (17) with $u_1 \neq 0$, and given $x_2 \neq x_1$ and $y_2 \neq y_1$, Theorem 4 shows that neither (6) nor (17) can be made exact at (x_2, y_2) if, and only if, either $(u_{21} = u_{11}$ and $u_{22} \neq u_{12})$ or $(u_{12} = u_{11}$ and $u_{22} \neq u_{21})$.

Our next theorem parallels the final part of Theorem 1 in describing 'best' upper bounds on $||v - u||$ when the multiplicative form (14) is used. Instead of using (17) we shall work directly with (14), taking $\min u(x, y) = r$ and $\max u(x, y) = r + 1$, and present the bounds as functions of r . The choice of r corresponds to the choice of K in (17) when, for example, u is fixed with $\min u = 0$ and $\max u = 1$. Because the midpoint $r + \frac{1}{2}$ of the interval for u equals 0 when $r = -1/2$, we consider only $r \geq -1/2$ explicitly.

THEOREM 5. Suppose v is specified by (14) with $\min u(x, y) = r$, $\max u(x, y) = r + 1$, and $u_{11} \neq 0$. Then:

(a) If $r \geq 0$, it is always possible to have $||v - u|| \leq \frac{3r^2 + 3r + 1}{2r^2 + 2r + 1}$, and the value of $u_{11} = u(x_1, y_1)$ that assures this bound is $u_{11} = (2r^2 + 2r + 1)/(2r + 1)$; if $-1/2 \leq r < 0$ then $||v - u|| \leq 1$ can be assured by taking $u_{11} = r + 1$;

(b) If u is monotonic in one variable and $r \geq 0$ then it is always possible to have $||v - u|| \leq 1$, and this bound is assured by every $u_{11} \in [(r+1)^2/(r+2), r+1]$; if u is monotonic in one variable and $-1/2 \leq r < 0$, then $||v - u|| \leq 1$ can be assured by taking $u_{11} = r+1$;

(c) If u is monotonic in both variables and $r \geq 0$ then it is always possible to have $||v - u|| \leq \frac{r+1}{2r+1}$, and this bound is assured by taking $u_{11} = r + \frac{1}{2}$; if u is monotonic and $-1/2 \leq r < 0$, then $||v - u|| \leq 1$ is assured by taking $u_{11} = r+1$.

This theorem and its ensuing proof show that monotonicity has no effect on the best general upper bound on $||v - u||$ when the image of u contains the origin and v is specified by (14). Moreover, the bounds in Theorem 5 are respectively smaller than, identical to, and larger than the additive approximation bounds from Theorem 1 for the three cases (a), (b) and (c). This is illustrated by Figure 3. As r increases, the bounds for multiplicative

Figure 3 about here

v approach the bounds for additive v .

Proof. Throughout this proof we write $E = |v - u|$, or

$$E = \left| \frac{u(x,y)u(x,y)}{u_{11}} - u(x,y) \right|.$$

Given $r \geq -1/2$, our objective is to identify a value of u_{11} between r and $r+1$ that ensures a 'best' upper bound on the value of E regardless of the nature of u so long as $\min u = r$, $\max u = r+1$, and u satisfies the monotonicity conditions (if any) that are specified.

If u is subject to no monotonicity restrictions and $r \geq 0$ then

$$\max_T E \leq \max_{u_{11}} \left\{ \frac{(r+1)^2}{u_{11}} - r, (r+1) - \frac{r^2}{u_{11}} \right\}.$$

The right hand side is minimized when $(r+1)^2/u_{11} - r = (r+1) - r^2/u_{11}$, or when $u_{11} = (2r^2 + 2r + 1)/(2r + 1)$, and this choice of u_{11} gives

$$\max E \leq \frac{3r^2 + 3r + 1}{2r^2 + 2r + 1},$$

as specified in Theorem 5(a). Continuing without monotonicity, suppose

$-1/2 \leq r < 0$. If $u_{11} > 0$ then

$$\max E \leq \max_{u_{11}} \left\{ \frac{(r+1)^2}{u_{11}} - r, \left| \frac{r(r+1)}{u_{11}} - (r+1) \right| \right\}.$$

The right hand side of this is minimized when u_{11} is maximized at $u_{11} = r+1$, so that $\max E \leq 1$, given $u_{11} > 0$. If $u_{11} < 0$ then

$$\max E \leq \max_{u_{11}} \left\{ \left| \frac{(r+1)^2}{u_{11}} - (r+1) \right|, \frac{r(r+1)}{u_{11}} - r \right\};$$

and, with $u_{11} = r$, $\max E \leq \max \{-(r+1)/r, 1\} = -(r+1)/r$. Since $-(r+1)/r \geq 1$ when $-1/2 \leq r < 0$, the best general result is achieved in this case by setting $u_{11} = r+1$, with $\max E \leq 1$. This completes the proof of Theorem 5(a)

For Theorem 5(b) assume without loss in generality that u is increasing in x . (If u decreases in x , a change in variable from x to $1-x$ leads to the same results.) An analysis of E according to the region of Figure 1(a) that contains (x,y) yields the following conclusions when $r \geq 0$:

- I. $u(x,y) \leq u(x_1,y), u(x,y_1) \leq u_{11}$. $\max E \leq \max \left\{ \frac{u_{11}(r+1)}{u_{11}} - r, \left| \frac{ru(x,y)}{u_{11}} - u(x,y) \right| \right\} = \max \{1, (r+1)(1 - r/u_{11})\} = 1;$
- II. $u(x_1,y) \leq u(x,y), u_{11} \leq u(x,y_1)$. $\max E \leq \max \{1, (r+1)[(r+1)/u_{11} - 1]\}$
- III. $u(x_1,y) \leq u(x,y), u_{11} \leq u(x,y_1)$. Same as II;
- IV. $u(x,y) \leq u(x_1,y), u(x,y_1) \leq u_{11}$. Same as I.

Therefore $\max E \leq \max \{1, (r+1)[(r+1)/u_{11} - 1]\} = 1$ whenever $(r+1)[(r+1)/u_{11} - 1] \leq 1$, i.e. whenever $u_{11} \geq (r+1)^2/(r+2)$, given $r \geq 0$. Suppose next that $-1/2 \leq r < 0$. Since this case is intermediate between (a) and (c) and, in each of these, $\max E \leq 1$ is obtained by taking $u_{11} = r+1$ (see below for (c)), the same result must hold for case (b).

For definiteness in (c) assume that u increases in both variables. Given $r \geq 0$, an analysis of E by the regions of Figure 1(a) yields the following:

- I. $u(x,y) \leq \{u(x_1,y), u(x,y_1)\} \leq u_{11}$. $\max E \leq \max \left\{ \frac{u_{11}}{u_{11}} - r, \max_{r \leq u(x,y) \leq u_{11}} \left| \frac{u(x,y)^2}{u_{11}} - u(x,y) \right| \right\} = u_{11} - r;$
- II. $u(x_1,y) \leq \{u(x,y), u_{11}\} \leq u(x,y_1)$. $\max E \leq \max \{(r+1)(1 - r/u_{11}), r+1 - u_{11}\}$, where the terms in braces are computed using $u(x,y) = u(x_1,y)$ and $u(x,y) = u(x,y_1)$;
- III. $u_{11} \leq \{u(x,y_1), u(x_1,y)\} \leq u(x,y)$. $\max E \leq \max \{|u_{11} - (r+1)|, \max_{u_{11} \leq u(x,y) \leq r+1} \left| \frac{u(x,y)^2}{u_{11}} - u(x,y) \right| \} = \max \{r+1 - u_{11}, \frac{r+1}{u_{11}}(r+1 - u_{11})\} = \frac{r+1}{u_{11}}(r+1 - u_{11});$
- IV. $u(x,y_1) \leq \{u(x,y), u_{11}\} \leq u(x_1,y)$. Same result as II.

Therefore, given $r \geq 0$,

$$\begin{aligned} \max E &\leq \max \left\{ u_{11} - r, \frac{r+1}{u_{11}}(u_{11} - r), r+1 - u_{11}, \frac{r+1}{u_{11}}(r+1 - u_{11}) \right\} \\ &= \max \left\{ \frac{r+1}{u_{11}}(u_{11} - r), \frac{r+1}{u_{11}}(r+1 - u_{11}) \right\} \\ &= (r+1) \max \left\{ 1 - \frac{r}{u_{11}}, \frac{r+1}{u_{11}} - 1 \right\}. \end{aligned}$$

The final expression here is minimized when the two terms in braces are equal, i.e. when $u_{11} = r + 1/2$, in which case $\max E \leq (r+1)/(2r+1)$. To complete the proof of (c), assume that $-1/2 \leq r < 0$. Given $u_{11} > 0$ a regional analysis of E shows that $\max E$ is governed by $(r+1)(u_{11} - r)/u_{11}$, whose minimum value equals 1 when $u_{11} = r+1$. A lower bound on E cannot be obtained by taking $u_{11} < 0$. Hence $\max E \leq 1$ with $u_{11} = r+1$ when $-1/2 \leq r < 0$. Q.E.D.

A final comment on the simple multiplicative form is in order before we consider another type of approximation. Suppose that X is utility independent of Y and Y is utility independent of X in the generalized sense discussed by Fishburn (1974) and Fishburn and Keeney (1974). Then, with $(x_0, y_0), (x_1, y_1) \in T$ such that $u_{11} \neq u_{01}$ and $u_{11} \neq u_{10}$, where $u_{01} = u(x_0, y_1)$ and $u_{10} = u(x_1, y_0)$ as in (18), it can be shown that $\|v - u\| = 0$ when v is defined by

$$v(x, y) = \frac{(u_{10}u_{01} - u_{00}u_{11})[u(x, y_1) + u(x_1, y) - u_{11}] + (u_{00} + u_{11} - u_{01} - u_{10})u(x, y)u(x_1, y)}{(u_{11} - u_{01})(u_{11} - u_{10})}. \quad (20)$$

If u is naturally additive then $u_{00} + u_{11} = u_{01} + u_{10}$ and (20) reduces to (6). If $u_{00} + u_{11} \neq u_{01} + u_{10}$ then (20) is identical to (17) when

$K = (u_{11}u_{00} - u_{01}u_{10})/[u_{11}(u_{11} - u_{10})(u_{11} - u_{01})]$, and this value of K is not equal to $-1/u_{11}$. This shows that, when u is not additive, the multiplicative form (14) or its counterpart (17) corresponds to the generalized version of Keeney's notion of utility independence in each direction.

The Simple Additive-Multiplicative Form

The simple additive and multiplicative approximations are exact along the two heavy line segments shown in Figure 1(a) when (x_1, y_1) is the fixed point used in the approximations. We shall now examine a mixed additive-multiplicative approximation that is based on two fixed points, (x_1, y_1) and (x_2, y_2) , and that is exact along the four heavy line segments of Figure 1(b). A different approximation that is also exact when $x \in \{x_1, x_2\}$ or $y \in \{y_1, y_2\}$ will be mentioned later in section 5.

THEOREM 6. Suppose that

$$v(x, y) = f_1(x) + g_2(y) + f_3(x)g_3(y), \text{ for all } (x, y) \in [0, 1]^2, \quad (21)$$

that $v(x, y) = u(x, y)$ when $x \in \{x_1, x_2\}$ or $y \in \{y_1, y_2\}$, and $u_{11} + u_{22} \neq u_{12} + u_{21}$. Then

$$v(x, y) = \frac{1}{u_{11} + u_{22} - u_{12} - u_{21}} \{ u(x_1, y)u(x, y_1) + u(x_2, y)u(x, y_2) - u(x_1, y)u(x, y_2) - u(x_2, y)u(x, y_1) + u(x_1, y)[u_{22} - u_{11}] + u(x_2, y)[u_{11} - u_{12}] + u(x, y_1)[u_{22} - u_{12}] + u(x, y_2)[u_{11} - u_{21}] + u_{12}u_{21} - u_{11}u_{22} \} \quad (22)$$

or, equivalently,

$$v(x,y) = u(x,y_1) + u(x_1,y) - u_{11} \quad (23)$$

$$+ \frac{[u(x,y_2) - u(x,y_1) + u_{11} - u_{12}][u(x_2,y) - u(x_1,y) + u_{11} - u_{21}]}{u_{11} + u_{22} - u_{12} - u_{21}},$$

and v is affine preserving.

Proof. Given (21) and the other initial conditions of the theorem, substitution of x_1 for x and then x_2 for x in (21) gives

$$g_2(y) + f_3(x_1)g_3(y) = u(x_1,y) - f_1(x_1),$$

$$g_2(y) + f_3(x_2)g_3(y) = u(x_2,y) - f_1(x_2).$$

Simultaneous solution of these equations for the 'unknowns' $g_2(y)$ and $g_3(y)$ gives

$$g_2(y) = \frac{f_3(x_2)[u(x_1,y) - f_1(x_1)] - f_3(x_1)[u(x_2,y) - f_1(x_2)]}{f_3(x_2) - f_3(x_1)},$$

$$g_3(y) = \frac{[u(x_2,y) - f_1(x_2)] - [u(x_1,y) - f_1(x_1)]}{f_3(x_2) - f_3(x_1)}.$$

Substitution of y_1 for y and then y_2 for y in (21) leads, in similar fashion, to

$$f_1(x) = \frac{g_3(y_2)[u(x,y_1) - g_2(y_1)] - g_3(y_1)[u(x,y_2) - g_2(y_2)]}{g_3(y_2) - g_3(y_1)}$$

$$f_3(x) = \frac{[u(x,y_2) - g_2(y_2)] - [u(x,y_1) - g_2(y_1)]}{g_3(y_2) - g_3(y_1)}.$$

In addition to these expressions for g_2 , g_3 , f_1 and f_3 , the given conditions require

$$\begin{aligned}
u_{11} &= f_1(x_1) + g_2(y_1) + f_3(x_1)g_3(y_1) \\
u_{12} &= f_1(x_1) + g_2(y_2) + f_3(x_1)g_3(y_2) \\
u_{21} &= f_1(x_2) + g_2(y_1) + f_3(x_2)g_3(y_1) \\
u_{22} &= f_1(x_2) + g_2(y_2) + f_3(x_2)g_3(y_2),
\end{aligned}$$

so that $[f_3(x_2) - f_3(x_1)][g_3(y_2) - g_3(y_1)] = u_{11} + u_{22} - u_{12} - u_{21}$, which is nonzero by presupposition. Hence the denominators of g_2 , g_3 , f_1 and f_3 do not vanish. By substituting the foregoing solutions for g_2 , g_3 , f_1 and f_3 into the right hand side of (21) and then using the displayed equations for u_{11} , u_{12} , u_{21} and u_{22} , it is readily verified that v is given by (22). The equivalence between (22) and (23) is most easily established by showing that the right hand side of (23) 'reduces' to the right hand side of (22). The form of v given by (23) shows immediately that v is affine preserving. Q.E.D.

Although (22) guarantees $|v - u| = 0$ for more line segments in T than does either (6) or (14), analysis of $||v - u||$ appears to be considerably more difficult for (22) than for the simple additive or multiplicative approximations. Because of this I shall discuss only one specific context for (22) that has interesting and easily derived properties. The form of (22) used in the ensuing theorem might be thought of as a boundary model or perimeter model since it guarantees that $|v(x,y) - u(x,y)| = 0$ whenever (x,y) lies on the boundary of I .

THEOREM 7. Suppose u is conservative [see (7)] and v is specified by (22) with $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,1)$. Then v is conservative also and $||v - u|| < \Delta/2$, where $\Delta = u(1,0) + u(0,1) - u(0,0) - u(1,1)$.

This theorem is directly comparable to Theorem 2 which uses the additive approximation. Theorem 7 does not guarantee a lower $||v - u||$ than does Theorem 2 (where $||v - u|| < \Delta/3$). However, $||v - u||$ for (22) can be considerably smaller than $\Delta/4$, and can equal zero, whereas $||v - u||$ must be at least as great as $\Delta/4$ when (6) is used for v . Moreover, the boundary model form of (22) has the attractive property of conservatism preservation.

Proof. The form of (22) specified in Theorem 7 can be written as

$$v(x,y) = \frac{1}{-\Delta} \{u(0,y)[u(x,0) - u(x,1) + u(1,1) - u(1,0)] + u(1,y)[u(x,1) - u(x,0) + u(0,0) - u(0,1)] + u(x,0)[u(1,1) - u(0,1)] + u(x,1)[u(0,0) - u(1,0)] + u(0,1)u(1,0) - u(0,0)u(1,1)\}.$$

Therefore, when $y' > y$,

$$\begin{aligned} v(x,y') - v(x,y) &= -\frac{1}{\Delta} \{[u(0,y') - u(0,y)][u(x,0) - u(x,1) + u(1,1) - u(1,0)] \\ &\quad + [u(1,y') - u(1,y)][u(x,1) - u(x,0) + u(0,0) - u(0,1)]\} \\ &> 0 \end{aligned}$$

since, under conservatism, $\Delta > 0$, $u(0,y') - u(0,y) > 0$, $u(1,y') - u(1,y) > 0$, and each of $u(x,0) - u(x,1) + u(1,1) - u(1,0)$ and $u(x,1) - u(x,0) + u(0,0) - u(0,1)$ is negative unless $x \in \{0,1\}$ in which case one of these terms is zero. Therefore v increases in y . Similarly, v increases in x .

To verify conservatism, or (7), for v , suppose $x < x'$ and $y < y'$. Then, after cancellations and rearrangement, we obtain

$$v(x,y') + v(x',y) - v(x,y) - v(x',y') = \frac{[u(0,y') - u(0,y) - u(1,y') + u(1,y)][u(x,0) - u(x,1) - u(x',0) + u(x',1)]}{-\Delta}$$

and this is positive since $[u(0,y') - u(0,y) - u(1,y') + u(1,y)] > 0$ and $[u(x,0) - u(x,1) - u(x',0) + u(x',1)] < 0$.

Finally, since both u and v are conservative and $v = u$ on the border of T , if $0 < x < 1$ and $0 < y < 1$ then

$$\begin{aligned} \max \{u(0,y) + u(x,1) - u(0,1), u(x,0) + u(1,y) - u(1,0)\} &< \min \{v(x,y), u(x,y)\} \\ &\leq \max \{v(x,y), u(x,y)\} < \min \{u(0,y) + u(x,0) - u(0,0), u(x,1) + u(1,y) - u(1,1)\}, \end{aligned}$$

so that

$$\begin{aligned} |v(x,y) - u(x,y)| &< \min \{u(x,0) - u(x,1) + u(0,1) - u(0,0), \\ &\quad u(x,1) - u(x,0) + u(1,0) - u(1,1), \\ &\quad u(0,y) - u(1,y) + u(1,0) - u(0,0), \\ &\quad u(1,y) - u(0,y) + u(0,1) - u(1,1)\}. \end{aligned} \quad (24)$$

The four terms in braces in (24) are positive under conservatism and their sum equals 2Δ . Hence the smallest of these four cannot exceed $\Delta/2$. Therefore $|v(x,y) - u(x,y)| < \Delta/2$ for all (x,y) . Q.E.D.

The following assertions, whose proofs are left to the reader, make additional connections with our previous discussion of conservatism. Let $R(x,y)$ equal the right hand side of (24), with v as in Theorem 7. Then $R(x,y)$ is uniquely maximized at the point (x^*,y^*) specified by (10) and (11), with $R(x^*,y^*) = \Delta/2$. $R(x,y)$ decreases on each ray out from (x^*,y^*) and is constant on the borders of rectangles whose corners lie on the $\varphi_{00} = \varphi_{11}$ and $\varphi_{01} = \varphi_{10}$ lines of Figure 2. Two such loci of constant R are identified by dashed lines on Figure 2.

4. LINEAR INTERPOLATIONS

The simple forms of (2) that were examined in the preceding section have each f_i and g_i as either the identity function or an expression based on conditional utility functions for one of the variables. In the present section we shall consider approximations by linear interpolation in which the functions involved in (2) that are not based on utility values are more complex than the identity function but nevertheless retain fairly simple forms. Needless to say, a vast array of nonlinear interpolation methods could be used to approximate $u(x,y)$, but, with the exception of a quasi-linear form that is mentioned below, we shall not go into these.

The approximations in this section are based on a set $\{x_1, \dots, x_p\}$ of $p \geq 2$ values of X and/or on a set $\{y_1, \dots, y_q\}$ of $q \geq 2$ values of Y , where

$$\begin{aligned} 0 &= x_1 < x_2 < \dots < x_p = 1 \\ 0 &= y_1 < y_2 < \dots < y_q = 1. \end{aligned} \quad (25)$$

Within the context of (25), we define the following nonnegative piecewise linear functions:

$$\alpha_i(x) = \begin{cases} \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (i = 1, \dots, p-1)$$

$$\alpha_i^*(x) = \begin{cases} 1 - \alpha_{i-1}(x) & \text{if } x_{i-1} \leq x \leq x_i \text{ and } i > 1 \\ \alpha_i(x) & \text{if } x_i \leq x \leq x_{i+1} \text{ and } i < p \\ 0 & \text{otherwise} \end{cases} \quad (i = 1, \dots, p);$$

$$\beta_j(y) = \begin{cases} \frac{y_{j+1} - y}{y_{j+1} - y_j} & \text{if } y_j \leq y \leq y_{j+1} \\ 0 & \text{otherwise} \end{cases} \quad (j = 1, \dots, q-1)$$

$$\beta_j^*(y) = \begin{cases} 1 - \beta_{j-1}(y) & \text{if } y_{j-1} \leq y \leq y_j \text{ and } j > 1 \\ \beta_j(y) & \text{if } y_j \leq y \leq y_{j+1} \text{ and } j < q \\ 0 & \text{otherwise} \end{cases} \quad (j = 1, \dots, q).$$

The α_i^* and β_j^* functions are continuous on $[0,1]$. For example, with $1 < i < p$, $\alpha_i^*(x)$ is zero up to x_{i-1} , increases linearly from 0 to 1 between x_{i-1} and x_i , decreases linearly from 1 to 0 between x_i and x_{i+1} , and is zero after x_{i+1} .

It is also useful to observe that

$$\alpha_i^*(x) + \alpha_{i+1}^*(x) = 1 \quad \text{on } [x_i, x_{i+1}] \quad \text{for } i = 1, \dots, p-1,$$

$$\beta_j^*(y) + \beta_{j+1}^*(y) = 1 \quad \text{on } [y_j, y_{j+1}] \quad \text{for } j = 1, \dots, q-1,$$

and, more generally, that $\sum_{i=1}^p \alpha_i^*(x) = \sum_{j=1}^q \beta_j^*(y) = 1$ for all x and y .

Four Models

Within the context of the foregoing definitions, we shall consider the following four approximations for u :

$$v_1(x, y) = \sum_{i=1}^p \alpha_i^*(x) u(x_i, y) \quad \text{for all } (x, y) \in T$$

$$v_2(x, y) = \sum_{j=1}^q \beta_j^*(y) u(x, y_j) \quad \text{for all } (x, y)$$

$$v_3(x, y) = v_1(x, y)/2 + v_2(x, y)/2 \quad \text{for all } (x, y) \in T,$$

$$\begin{aligned}
v_4(x,y) = & \alpha_i(x)\beta_j(y)u_{ij} + \alpha_i(x)[1 - \beta_j(y)]u_{i,j+1} + [1 - \alpha_i(x)]\beta_j(y)u_{i+1,j} \\
& + [1 - \alpha_i(x)][1 - \beta_j(y)]u_{i+1,j+1} \quad \text{for all } (x,y) \in [x_i, x_{i+1}] \times \\
& [y_j, y_{j+1}], \text{ with } i = 1, \dots, p-1; \\
& j = 1, \dots, q-1,
\end{aligned}$$

where $u_{ij} = u(x_i, y_j)$ as in (18). The points in T at which these approximations are exact are illustrated by Figure 4(a): $v_1 = u$ along the vertical

Figure 4 about here

lines; $v_2 = u$ along the horizontal lines; and $v_3 = v_4 = u$ at the pq lattice points $(x_1, y_1), (x_1, y_2), \dots, (x_p, y_q)$. These assertions follow immediately from the definitions and the observation that

$$v_1(x,y) = \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)u(x_i, y) + \left(\frac{x - x_i}{x_{i+1} - x_i}\right)u(x_{i+1}, y) \quad \text{if } x_i \leq x \leq x_{i+1} \quad (26)$$

$$v_2(x,y) = \left(\frac{y_{j+1} - y}{y_{j+1} - y_j}\right)u(x, y_j) + \left(\frac{y - y_j}{y_{j+1} - y_j}\right)u(x, y_{j+1}) \quad \text{if } y_j \leq y \leq y_{j+1}. \quad (27)$$

The determinations of $v(x,y)$ for $(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ are noted with reference to Figure 4(b). The value of $v_1(x,y)$ is the convex linear combination of $u(x_i, y)$ and $u(x_{i+1}, y)$ shown by (26); $v_2(x,y)$ is the linearly interpolated value between $u(x, y_j)$ and $u(x, y_{j+1})$ shown by (27); $v_3(x,y)$ is the average of the horizontal and vertical interpolations given by $v_1(x,y)$ and $v_2(x,y)$ [modification of v_3 to the form $\lambda v_1 + (1 - \lambda)v_2$ allows different emphases to be placed on the horizontal and vertical interpolations]; and $v_4(x,y)$ is given by a weighted average of the four u values at the corners of $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. The quasi-linear approximation v_4 can be thought

of as a two-stage linear interpolation process since, with $\alpha = \alpha_1(x)$ and $\beta = \beta_j(y)$,

$$\begin{aligned} v_4(x,y) &= \alpha v_4(x_1,y) + (1-\alpha)v_4(x_{i+1},y) \\ &= \alpha[\beta u_{1j} + (1-\beta)u_{1,j+1}] + (1-\alpha)[\beta u_{i+1,j} + (1-\beta)u_{i+1,j+1}] \\ &= \beta[\alpha u_{1j} + (1-\alpha)u_{i+1,j}] + (1-\beta)[\alpha u_{1,j+1} + (1-\alpha)u_{i+1,j+1}] \\ &= \beta v_4(x,y_j) + (1-\beta)v_4(x,y_{j+1}). \end{aligned}$$

Moreover, if u is linear along each of the four border line segments of the rectangle $[x_1, x_{i+1}] \times [y_j, y_{j+1}]$, then $v_3 = v_4$ throughout the rectangle, but, without such linearity, coincidence of v_3 and v_4 is assured only at the four corners.

There are several major differences between v_4 and the other three approximations. First, v_1 , v_2 and v_3 presume that p , q or $p+q$ conditional utility functions are evaluated, whereas v_4 uses only the u values at the pq lattice points. Secondly, v_1 , v_2 and v_3 are clearly within the format of (2), whereas this is not at all clear for v_4 . I leave it to the reader to show that v_4 can indeed be viewed as a special case of (2), but it appears that this can be done only when some of the f_1 and g_1 functions have discontinuities. As shown by the definitions of v_1 , v_2 and v_3 , and by previous discussion, the f_1 and g_1 functions used therein are all continuous on $[0,1]$.

Despite these differences, all four approximations possess certain preservation properties defined earlier.

THEOREM 8. Each of v_1 , v_2 , v_3 and v_4 is continuous, affine preserving, monotonicity preserving, and conservatism preserving.

Proof. Because of the similarities among v_1 , v_2 , and v_3 , it will suffice to consider only v_1 and v_4 . Continuity for v_1 is obvious and it is easily seen to hold for v_4 . Affine preservation is easily checked. For monotonicity suppose first that $x_1 \leq x < x^* \leq x_{i+1}$ and that u is monotonic in x . Then, by (26),

$$v_1(x^*, y) - v_1(x, y) = \left(\frac{x^* - x}{x_{i+1} - x_1} \right) [u(x_{i+1}, y) - u(x_1, y)], \quad (28)$$

so that $v_1(x^*, y) - v_1(x, y)$ has the same sign as $u(x_{i+1}, y) - u(x_1, y)$. Hence v_1 is monotonic in x on the interval $[x_1, x_{i+1}]$ in the same sense that u is monotonic in x on this interval. Since this is true for all $[x_1, x_{i+1}]$, v_1 is monotonicity preserving in x . Suppose next that $y_j \leq y < y^* \leq y_{j+1}$ and that u is monotonic in y . Then, by (26),

$$\begin{aligned} v_1(x, y^*) - v_1(x, y) &= \left(\frac{x_{i+1} - x}{x_{i+1} - x_1} \right) [u(x_1, y^*) - u(x_1, y)] \\ &\quad + \left(\frac{x - x_1}{x_{i+1} - x_1} \right) [u(x_{i+1}, y^*) - u(x_{i+1}, y)], \end{aligned}$$

from which it follows that v_1 is monotonicity preserving in y . With regard to v_4 , if $x_1 \leq x < x^* \leq x_{i+1}$ and $y \in [y_j, y_{j+1}]$ then

$$\begin{aligned} v_4(x^*, y) - v_4(x, y) &= [\alpha_1(x) - \alpha_1(x^*)] [\beta_j(y)(u_{i+1,j} - u_{1j}) \\ &\quad + (1 - \beta_j(y))(u_{i+1,j+1} - u_{1,j+1})] \end{aligned} \quad (29)$$

with $\alpha_1(x) > \alpha_1(x^*)$. Hence if u is monotonic in x then v_4 is monotonic in x in the same sense as u on the interval from x_1 to x_{i+1} and, since this is true of each such interval, v_4 is monotonicity preserving in x . The proof that v_4 is monotonicity preserving in y is similar.

To establish conservatism preservation we work first within a rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $x_i \leq x < x^* \leq x_{i+1}$ and $y_j \leq y < y^* \leq y_{j+1}$, and assume that u is conservative [see (7)]. Using (28),

$$\begin{aligned} v_1(x, y^*) + v_1(x^*, y) - v_1(x, y) - v_1(x^*, y^*) &= \left(\frac{x^* - x}{x_{i+1} - x_i} \right) [u(x_i, y^*) + u(x_{i+1}, y) \\ &\quad - u(x_i, y) - u(x_{i+1}, y^*)]. \end{aligned}$$

Since conservatism for u implies that the right hand side of this equation is positive, it follows that v_1 is conservative in the rectangle. Using (29),

$$\begin{aligned} v_4(x, y^*) + v_4(x^*, y) - v_4(x, y) - v_4(x^*, y^*) &= [\alpha_1(x) - \alpha_1(x^*)][\beta_j(y) - \beta_j(y^*)] \\ &\quad [u_{1,j+1} + u_{i+1,j} - u_{1j} \\ &\quad - u_{i+1,j+1}], \end{aligned}$$

and, since each term on the right hand side is positive, the left side is positive also. Hence v_4 is conservative in the rectangle. Therefore both v_1 and v_4 are conservative in every rectangle of the form $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. It then follows without difficulty--by breaking any rectangle $[x, x^*] \times [y, y^*]$ into subrectangles according to the grid of Figure 4(a), and adding up the inequalities implied by conservatism on each of the subrectangles--that v_1 and v_4 are conservative throughout T . Q.E.D.

Uniform Norm Considerations

Given (25) and $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, the absolute differences between $u(x, y)$ and the approximations defined above are as follows:

$$|v_1(x,y) - u(x,y)| = |\alpha_1(x)[u(x_1,y) - u(x,y)] + (1 - \alpha_1(x))[u(x_{1+1},y) - u(x,y)]|$$

$$|v_2(x,y) - u(x,y)| = |\beta_j(y)[u(x,y_j) - u(x,y)] + (1 - \beta_j(y))[u(x,y_{j+1}) - u(x,y)]|$$

$$|v_3(x,y) - u(x,y)| = \frac{1}{2}|\alpha_1(x)[u(x_1,y) - u(x,y)] + (1 - \alpha_1(x))[u(x_{1+1},y) - u(x,y)] + \beta_j(y)[u(x,y_j) - u(x,y)] + (1 - \beta_j(y))[u(x,y_{j+1}) - u(x,y)]|$$

$$|v_4(x,y) - u(x,y)| = |\alpha_1(x)\beta_j(y)[u_{1j} - u(x,y)] + \alpha_1(x)(1 - \beta_j(y))[u_{1,j+1} - u(x,y)] + (1 - \alpha_1(x))\beta_j(y)[u_{1+1,j} - u(x,y)] + (1 - \alpha_1(x))(1 - \beta_j(y))[u_{1+1,j+1} - u(x,y)]|.$$

In the present setting it is natural to consider the effects of increases in p and/or q on $\|v - u\|$. Although this can be done with either equal or unequal spacing of the x_i or y_j in $[0,1]$, I shall consider equal spacing for expository simplicity. Thus, let $v_1^{(p)}$ denote v_1 when the p points in (25) are equally spaced in $[0,1]$, with $x_1 = (i-1)/(p-1)$ and $x_{i+1} - x_i = 1/(p-1)$; let $v_2^{(q)}$ denote v_2 under similar convention; let $v_3^{(p)} = v_3^{(p)}/2 + v_2^{(p)}/2$; and let $v_4^{(p)}$ denote v_4 when $p = q$ in (25) with $x_i = (i-1)/(p-1)$ and $y_j = (j-1)/(p-1)$.

We shall now observe that each $\|v_k^{(p)} - u\|$ for $k = 1, 2, 3, 4$ approaches zero as p gets large. This will be done using moduli of continuity, which are measures of the variations of continuous functions on compact sets. With respect to the utility function u on T , we define

$$\omega(u, h) = \max \{ |u(x, y) - u(x', y')| : x, x', y, y' \in [0, 1], |x - x'| \leq h \text{ and } |y - y'| \leq h \},$$

$$\omega_1(u, h) = \max \{ |u(x, y) - u(x', y)| : x, x', y \in [0, 1], |x - x'| \leq h \},$$

$$\omega_2(u, h) = \max \{ |u(x, y) - u(x, y')| : x, y, y' \in [0, 1], |y - y'| \leq h \},$$

for all $h \in [0,1]$. Considered as functions of h , ω is the modulus of continuity for u , and ω_1 and ω_2 are partial moduli of continuity for u . These definitions presume nothing about differentiability or monotonicity for u . It is easily seen that the moduli are nonnegative, nondecreasing and satisfy

$$\max \{ \omega_1(u,h), \omega_2(u,h) \} \leq \omega(u,h) \leq \omega_1(u,h) + \omega_2(u,h).$$

Moreover, because u is continuous on a closed and bounded set, it follows readily from standard results (e.g., Bolzano-Weierstrass theorem, existence of convergent subsequences) that each of $\omega(u,h)$, $\omega_1(u,h)$ and $\omega_2(u,h)$ approaches zero as $h \rightarrow 0$.

THEOREM 9. $\|v_k^{(p)} - u\| \rightarrow 0$ as $p \rightarrow \infty$ for each $k \in \{1,2,3,4\}$. In particular,

$$\|v_k^{(p)} - u\| \leq \omega_k(u, 1/(p-1)) \quad \text{for } k = 1, 2,$$

$$\|v_3^{(p)} - u\| \leq [\omega_1(u, 1/(p-1)) + \omega_2(u, 1/(p-1))]/2 \leq \omega(u, 1/(p-1)),$$

$$\|v_4^{(p)} - u\| \leq \omega(u, 1/(p-1)),$$

and if each of ω , ω_1 and ω_2 is a concave function of h then

$$\|v_k^{(p)} - u\| \leq \omega_k(u, \frac{1}{2(p-1)}) \quad \text{for } k = 1, 2,$$

$$\|v_3^{(p)} - u\| \leq [\omega_1(u, \frac{1}{2(p-1)}) + \omega_2(u, \frac{1}{2(p-1)})]/2 \leq \omega(u, \frac{1}{2(p-1)}),$$

$$\|v_4^{(p)} - u\| \leq \omega(u, \frac{9}{16(p-1)}).$$

Proof. The initial results of the theorem follow readily from the expressions for $|v_k(x,y) - u(x,y)|$ written earlier and from the foregoing comments on the moduli of continuity. For example, omitting u from ω for notational convenience,

$$\begin{aligned}
|v_1^{(p)}(x,y) - u(x,y)| &\leq \alpha_1(x) |u(x_1,y) - u(x,y)| + (1 - \alpha_1(x)) |u(x_{j+1},y) - u(x,y)| \\
&\leq \alpha_1(x) \omega_1([1 - \alpha_1(x)](x_{j+1} - x_1)) + (1 - \alpha_1(x)) \\
&\quad \omega_1(\alpha_1(x)(x_{j+1} - x_1)) \\
&= \alpha_1(x) \omega_1\left(\frac{1 - \alpha_1(x)}{p - 1}\right) + (1 - \alpha_1(x)) \omega_1\left(\frac{\alpha_1(x)}{p - 1}\right),
\end{aligned}$$

so that

$$\begin{aligned}
||v_1^{(p)} - u|| &\leq \max_{0 \leq \alpha \leq 1} [\alpha \omega_1\left(\frac{1 - \alpha}{p - 1}\right) + (1 - \alpha) \omega_1\left(\frac{\alpha}{p - 1}\right)] \\
&\leq \max_{\alpha} [\alpha \omega_1\left(\frac{1}{p - 1}\right) + (1 - \alpha) \omega_1\left(\frac{1}{p - 1}\right)] \\
&= \omega_1(1/(p - 1)).
\end{aligned}$$

If ω_1 is concave in h then $\alpha \omega_1((1 - \alpha)/(p - 1)) + (1 - \alpha) \omega_1(\alpha/(p - 1)) \leq \omega_1([\alpha(1 - \alpha) + (1 - \alpha)\alpha]/(p - 1)) \leq \omega_1([1/2]/(p - 1))$. To verify the concavity result for v_u , we note first that

$$\begin{aligned}
|v_u^{(p)}(x,y) - u(x,y)| &\leq \alpha_1(x) \beta_j(y) |u_{1j} - u(x,y)| + \alpha_1(x) (1 - \beta_j(y)) |u_{1,j+1} - u(x,y)| \\
&\quad - u(x,y)| + (1 - \alpha_1(x)) \beta_j(y) |u_{1+1,j} - u(x,y)| \\
&\quad + (1 - \alpha_1(x)) (1 - \beta_j(y)) |u_{1+1,j+1} - u(x,y)| \\
&\leq \alpha_1(x) \beta_j(y) \omega(\max\{1 - \alpha_1(x), 1 - \beta_j(y)\}/(p - 1)) \\
&\quad + \alpha_1(x) (1 - \beta_j(y)) \omega(\max\{1 - \alpha_1(x), \beta_j(y)\}/(p - 1)) \\
&\quad + (1 - \alpha_1(x)) \beta_j(y) \omega(\max\{\alpha_1(x), 1 - \beta_j(y)\}/(p - 1)) \\
&\quad + (1 - \alpha_1(x)) (1 - \beta_j(y)) \omega(\max\{\alpha_1(x), \beta_j(y)\}/(p - 1)).
\end{aligned}$$

Therefore

$$\begin{aligned}
||v_4^{(p)} - u|| \leq \max_{\substack{0 < \alpha < 1 \\ 0 < \beta < 1}} & [\alpha\beta\omega(\max\{1-\alpha, 1-\beta\}/(p-1)) \\
& + \alpha(1-\beta)\omega(\max\{1-\alpha, \beta\}/(p-1)) \\
& + (1-\alpha)\beta\omega(\max\{\alpha, 1-\beta\}/(p-1)) \\
& + (1-\alpha)(1-\beta)\omega(\max\{\alpha, \beta\}/(p-1))].
\end{aligned}$$

Suppose that $\alpha \leq \beta \leq 1 - \alpha$. Then, assuming concavity for ω ,

$$\begin{aligned}
||v_4^{(p)} - u|| &\leq \max_{\alpha, \beta} [\alpha\beta\omega(\frac{1-\alpha}{p-1}) + \alpha(1-\beta)\omega(\frac{1-\alpha}{p-1}) \\
&\quad + (1-\alpha)\beta\omega(\frac{1-\beta}{p-1}) + (1-\alpha)(1-\beta)\omega(\frac{\beta}{p-1})] \\
&\leq \max_{\alpha, \beta} [\alpha\omega(\frac{1-\alpha}{p-1}) + (1-\alpha)\omega(\frac{2\beta(1-\beta)}{p-1})] \\
&\leq \max_{\alpha, \beta} \omega(\frac{(1-\alpha)[\alpha + 2\beta(1-\beta)]}{p-1}) \\
&= \omega(\frac{9/16}{p-1})
\end{aligned}$$

where the maximizing values are $\alpha = 1/4$ and $\beta = 1/2$. Because of the symmetry in α and $1 - \alpha$, and in β and $1 - \beta$, this suffices to establish the final conclusion of the theorem. Q.E.D.

Although Theorem 9 might be used in practical situations to estimate an upper bound on the maximum difference between $v(x, y)$ and $u(x, y)$ for a given p that can be assured by a linear model, it says very little about good ways to choose the x_i and/or y_j for (25) in attempting to minimize $||v - u||$ for fixed p and/or q .

To illustrate the latter idea, suppose u increases in both variables and v_4 is used with $p = q = 3$. Then the best general assertion that can be made for an upper bound on $||v_4 - u||$ is

$$||v_4 - u|| \leq \min_{\substack{0 \leq x_2 \leq 1 \\ 0 \leq y_2 \leq 1}} [\max \{u(x_2, 1) - u(0, y_2), u(1, y_2) - u(x_2, 0), \\ u(1, 1) - u(x_2, y_2), u(x_2, y_2) - u(0, 0)\}]. \quad (30)$$

For convenience set $u(0, 0) = 0$ and $u(1, 1) = 1$, and let V denote the value of the right hand side of (30). Then, because of $1 - u(x_2, y_2)$ and $u(x_2, y_2) - 0$ in (30), $V \geq 1/2$. In any event, it can be shown that some point that satisfies $u(x_2, 1) - u(0, y_2) = u(1, y_2) - u(x_2, 0)$, or $u(x_2, 0) + u(x_2, 1) = u(0, y_2) + u(1, y_2)$, must be a minimaxing point for V . There exists a unique (x_2^*, y_2^*) that satisfies $u(x_2^*, 0) + u(x_2^*, 1) = u(0, y_2^*) + u(1, y_2^*)$ and $u(x_2^*, y_2^*) = 1/2$. Consequently, if $u(x_2^*, 1) - u(0, y_2^*) \leq 1/2$ then $V = 1/2$, and if $u(x_2^*, 1) - u(0, y_2^*) > 1/2$ then $V > 1/2$. In the latter case, (x_2^*, y_2^*) may or may not be a minimaxing point for V , depending on the behavior of $u(x_2, 1) - u(0, y_2)$ relative to $u(x_2, y_2)$ along the curve through T that gives the (x_2, y_2) solutions to $u(x_2, 0) + u(x_2, 1) = u(0, y_2) + u(1, y_2)$. However, the choice of (x_2^*, y_2^*) as the interior point to use for v_4 when $p = q = 3$ appears to be reasonable.

5. EXACT GRID MODELS

In concluding our discussion of approximations for u on $T = [0, 1]^2$ we shall consider several approximations that are exact ($v = u$) on both the horizontal and vertical line segments of a grid on T such as shown in Figure 4(a). To focus the discussion we shall say that an approximation v is an exact grid model if $v(x_i, y) = u(x_i, y)$ for all $y \in [0, 1]$ for at least two distinct x_i , and if $v(x, y_j) = u(x, y_j)$ for all $x \in [0, 1]$ for at least two distinct y_j .

The only approximation of previous sections that is an exact grid model is the simple additive-multiplicative model discussed at the end of section 3. Approximations v_3 and v_4 of the preceding section are not exact grid models since they are exact only at the lattice points or intersection points of the grid. On the other hand, the simple forms of section 3 can all be adapted to serve as exact grid models on any grid formed from a finite number of horizontal and vertical lines through T by applying these forms in a patchwork or cut-and-paste fashion to different sections of the grid. Suppose, for example, that (25) holds with $p \geq 3$ and $q \geq 3$. Then, provided that $u_{ij} + u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} \neq 0$ for each $(i,j) \in \{1, \dots, p-1\} \times \{1, \dots, q-1\}$, the simple additive-multiplicative form can be applied separately to each rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. That is, (23) with x_1, x_2, y_1, y_2 replaced by $x_i, x_{i+1}, y_j, y_{j+1}$ is taken to hold throughout $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $i = 1, \dots, p-1$ and $j = 1, \dots, q-1$. The resultant approximation is exact on the grid of Figure 4(a).

Alternatively, with $0 = x_1 < x'_1 < x_2 < x'_2 < \dots < x_{p-1} < x'_{p-1} < x_p = 1$ and $0 = y_1 < y'_1 < y_2 < y'_2 < \dots < y_{q-1} < y'_{q-1} < y_q = 1$, one could use the additive form

$$v(x,y) = u(x,y'_j) + u(x'_i,y) - u(x'_i,y'_j) \text{ for all } (x,y) \in [x_i, x_{i+1}) \times [y_j, y_{j+1}),$$

except that $[x_i, x_{i+1})$ is replaced by $[x_i, x_{i+1}]$ when $i = p-1$, and $[y_j, y_{j+1})$ is replaced by $[y_j, y_{j+1}]$ when $j = q-1$. The resultant approximation is exact on the grid determined by $x'_1, x'_2, \dots, x'_{p-1}$ and $y'_1, y'_2, \dots, y'_{q-1}$, but it has one serious disadvantage that is not shared by the patchwork adaptation of the simple additive-multiplicative model based on (23), and that is its propensity for discontinuities along the x_i and y_j lines. A similar

disadvantage arises with an adaptation of the simple multiplicative model. Hence, because of the analytical difficulties that accompany discontinuities, the only one of the three simple forms of section 3 that appears to adapt itself reasonably well to a patchwork format is the additive-multiplicative form. The adaptation of this form is continuous, and Theorem 7, involving conservatism, is easily generalized to the patchwork format.

A Generalized Multiplicative Form

In the remainder of this section we shall consider a generalization of the simple multiplicative form that is not a patchwork adaptation. It is based on m fixed points for X and for Y subject to

$$\begin{aligned} 0 \leq x_1 < x_2 < \dots < x_m \leq 1 \\ 0 \leq y_1 < y_2 < \dots < y_m \leq 1, \end{aligned} \quad (31)$$

and is derived directly from (2) and the restriction that $v = u$ along each line determined by the $2m$ fixed points in (31). Its basic form is given as

$$v(x,y) = \sum_{i=1}^m \sum_{j=1}^m c_{ij} u(x,y_j) u(x_i,y), \quad (32)$$

where the c_{ij} are based on the m -by- m matrix of u_{ij} values ($i,j = 1, \dots, m$). When $m = 1$, $c_{11} = 1/u_{11}$ and (32) reduces to the simple multiplicative model (14). When $m = 2$, $c_{11} = u_{22}/A$, $c_{12} = -u_{21}/A$, $c_{21} = -u_{12}/A$, and $c_{22} = u_{11}/A$, where $A = u_{11}u_{22} - u_{12}u_{21}$, the determinant of the u_{ij} matrix. Thus the $m = 2$ version of (32) provides an alternative to the exact 2-by-2 grid model of the simple additive-multiplicative form so long as the u_{ij} matrix is nonsingular.

Approximation (32) seems attractive for several reasons. It has a nice analytical form, is exact on the grid determined by the points in (31), and is continuous. Moreover, if u is differentiable then v is differentiable. However, it is neither affine preserving nor monotonicity preserving, and it does not submit easily to analyses of $\|v - u\|$. Although very little is known about $\|v - u\|$ at the present time, it is hoped that further research will determine the conditions under which (32) gives a good approximation to u .

Because of the absence of interesting results on $\|v - u\|$, I shall present only the basic derivation of (32) from (2) and note the effects of positive affine transformations on this approximation. This presentation parallels the discussion for the simple multiplicative form in (14) through (17). In the present setting we shall presume that $m \geq 2$ and, given u and (31), let U denote the m -by- m matrix $[u_{ij}]$. Also let U_{ij} be the $(m-1)$ -by- $(m-1)$ matrix obtained from U by deleting its i th row and j th column and, with \det the determinant function on square matrices, define

$$A = \det(U)$$

$$A_{ij} = (-1)^{i+j} \det(U_{ij}) \quad i, j = 1, \dots, m,$$

so that A_{ij} is the cofactor of u_{ij} .

THEOREM 10. Let u and (31) be given with $m \geq 2$, and suppose that

$$v(x, y) = \sum_{k=1}^m f_k(x) g_k(y) \quad \text{for all } (x, y) \in [0, 1]^2 \quad (33)$$

with $v(x, y) = u(x, y)$ whenever $x \in \{x_1, \dots, x_m\}$ or $y \in \{y_1, \dots, y_m\}$, and that U is nonsingular ($A \neq 0$). Then

$$v(x,y) = \sum_{i=1}^m \sum_{j=1}^m (A_{ij}/A) u(x,y_j) u(x_i,y) \quad \text{for all } (x,y) \in [0,1]^2. \quad (34)$$

Moreover, if a and b are real numbers with $a > 0$, and if $aA + b \sum_{i=1}^m \sum_{j=1}^m A_{ij} \neq 0$, then

$$\frac{v^{ab}(x,y) - b}{a} = v(x,y) + K(a,b) \left[A - \sum_{i=1}^m \sum_{j=1}^m A_{ij} u(x_i,y) \right] \quad (35)$$

$$\left[A - \sum_{i=1}^m \sum_{j=1}^m A_{ij} u(x,y_j) \right]$$

where

$$K(a,b) = \frac{-b}{A[aA + b \sum_{i=1}^m \sum_{j=1}^m A_{ij}]}. \quad (36)$$

Expression (35) is the appropriate generalization of (16) or (17) for $m = 1$. As seen by (36), the only real value of K that cannot be obtained with admissible values of a and b is $K = -1/(A \sum_{ij} A_{ij})$, corresponding to $a = 0$, provided that $\sum_{ij} A_{ij} \neq 0$. If $\sum_{ij} A_{ij} = 0$ then every real K is admissible. Moreover, if $b \neq 0$, then (35) shows that $v^{ab}(x,y) = av(x,y) + b$ if and only if $[A - \sum_{ij} A_{ij} u(x_i,y)][A - \sum_{ij} A_{ij} u(x,y_j)] = 0$, which holds if $x \in \{x_1, \dots, x_m\}$ or $y \in \{y_1, \dots, y_m\}$ since $\sum_i A_{ij} u_{ij} = \sum_j A_{ij} u_{ij} = A$ and $\sum_i A_{ih} u_{ij} = 0$ if $h \neq j$ and $\sum_j A_{kj} u_{ij} = 0$ if $k \neq i$, but which cannot be expected to hold otherwise.

We conclude with an outline of the proof of Theorem 10 since the complete proof is rather long. Assume henceforth that the first sentence of the theorem applies. Substitution of x_i and then y_j into (33) gives

$$\sum_{k=1}^m f_k(x_i) g_k(y) = u(x_i, y) \quad i = 1, \dots, m$$

$$\sum_{k=1}^m f_k(x) g_k(y_j) = u(x, y_j) \quad j = 1, \dots, m.$$

Let F be the m -by- m matrix whose entry in row i and column k is $f_k(x_i)$, and let G be the m -by- m matrix whose entry in row j and column k is $g_k(y_j)$. F_{ik} and G_{jk} are defined similarly to U_{ij} by deleting rows and columns. Then, if F and G are nonsingular, Cramer's rule gives

$$g_k(y) = \sum_{i=1}^m u(x_i, y) (-1)^{i+k} \det(F_{ik}) / \det(F) \quad k = 1, \dots, m$$

$$f_k(x) = \sum_{j=1}^m u(x, y_j) (-1)^{j+k} \det(G_{jk}) / \det(G) \quad k = 1, \dots, m.$$

It then follows from the product and transposition rules for determinants (with a prime denoting transposition) that $\det(F) \det(G) = \det(F) \det(G') = \det(FG') = \det([\sum_k f_k(x_i) g_k(y_j)]) = \det(U) = A$, since the initial conditions require $u_{ij} = \sum_k f_k(x_i) g_k(y_j)$, and hence that neither F nor G is singular with

$$\sum_{k=1}^m f_k(x) g_k(y) = \frac{1}{A} \sum_{i=1}^m \sum_{j=1}^m u(x, y_j) u(x_i, y) (-1)^{i+j} \sum_{k=1}^m \det(F_{ik} G'_{jk}). \quad (37)$$

It can be shown that $\det(U_{ij}) = \sum_k \det(F_{ik} G'_{jk})$, and therefore (33) and (37) yield (34).

Assume henceforth that (34) holds and that $aA + b \sum_{ij} A_{ij} \neq 0$ with $a > 0$. Let $U^{ab} = [au_{ij} + b]$ with $A^{ab} = \det(U^{ab})$ and A_{ij}^{ab} the cofactor of $au_{ij} + b$ in U^{ab} . Then direct substitution in (34) gives

$$v^{ab}(x,y) = \sum_{i=1}^m \sum_{j=1}^m (A_{ij}^{ab}/A^{ab}) [au(x,y_j) + b][au(x_i,y) + b]. \quad (38)$$

Let σ denote a generic permutation on $\{1, \dots, m\}$, let $\text{sgn}(\sigma)$ be the number of inversions in σ (an inversion occurs when $i < j$ and $\sigma_i > \sigma_j$), and let $\pi(i,k;\sigma)$ be the product of the $m-2$ $u_{j\sigma_j}$ for those $j \notin \{i,k\}$. It can then be shown that

$$A^{ab} = a^m A + a^{m-1} b \sum_{i=1}^m \sum_{j=1}^m A_{ij}, \quad (39)$$

$$A_{ij}^{ab} = a^{m-1} A_{ij} + a^{m-2} b \sum_{\{\sigma: \sigma_i=j\}} (-1)^{\text{sgn}(\sigma)} \sum_{\{k:k \neq i\}} \pi(i,k;\sigma), \quad (40)$$

and that

$$\begin{aligned} A \sum_{\{\sigma: \sigma_i=j\}} (-1)^{\text{sgn}(\sigma)} \sum_{\{k:k \neq i\}} \pi(i,k;\sigma) &= A_{ij} \left(\sum_{k=1}^m \sum_{h=1}^m A_{kh} \right) \\ &\quad - \left(\sum_{h=1}^m A_{ih} \right) \left(\sum_{k=1}^m A_{kj} \right). \end{aligned} \quad (41)$$

The last identity allows one to conclude immediately that the sum of its left hand side over i equals zero and that the sum of its left hand side over j equals zero. This fact along with substitution of (40) and (41) in (38) gives

$$\begin{aligned} A^{ab} v^{ab}(x,y) &= \sum_{ij} [a^{m-1} A_{ij} + a^{m-2} b \sum_{\{\sigma: \sigma_i=j\}} (-1)^{\text{sgn}(\sigma)} \sum_{\{k:k \neq i\}} \pi(i,k;\sigma)] [a^2 u(x,y_j)u(x_i,y) \\ &\quad + ab(u(x,y_j) + u(x_i,y)) + b^2] \\ &= a^{m-1} \sum_{ij} A_{ij} u(x,y_j)u(x_i,y) + a^m b \sum_{ij} A_{ij} (u(x,y_j) + u(x_i,y)) \\ &\quad + a^{m-2} b^2 \sum_{ij} A_{ij} + a^m b \sum_{ij} u(x,y_j)u(x_i,y) [A_{ij} \sum_{kh} A_{kh} \\ &\quad - (\sum_h A_{ih})(\sum_k A_{kj})] / A. \end{aligned}$$

Multiplication by A and the use of (34) then gives

$$\begin{aligned}
 AA^{ab} v^{ab}(x,y) &= a^{m+1} A^2 v(x,y) + a^m bA[\sum_{ij} A_{ij} u(x,y_j) + \sum_{ij} A_{ij} u(x_i,y)] \\
 &\quad + a^{m-1} b^2 A \sum_{ij} A_{ij} + a^m bA \sum_{ij} A_{ij} v(x,y) \\
 &\quad - a^m b(\sum_{ij} A_{ij} u(x,y_j))(\sum_{ij} A_{ij} u(x_i,y)) + a^m bA^2 - a^m bA \\
 &= bA[a^m A + a^{m-1} b \sum_{ij} A_{ij}] + av(x,y)A[a^m A + a^{m-1} b \sum_{ij} A_{ij}] \\
 &\quad - a^m b[A^2 - A(\sum_{ij} A_{ij} u(x,y_j) + \sum_{ij} A_{ij} u(x_i,y))] \\
 &\quad + (\sum_{ij} A_{ij} u(x,y_j))(\sum_{ij} A_{ij} u(x_i,y))
 \end{aligned}$$

which, since $AA^{ab} = A[a^m A + a^{m-1} b \sum_{ij} A_{ij}]$ by (39), yields

$$v^{ab}(x,y) = av(x,y) + b - \frac{a^m b[A - \sum_{ij} A_{ij} u(x,y_j)][A - \sum_{ij} A_{ij} u(x_i,y)]}{A[a^m A + a^{m-1} b \sum_{ij} A_{ij}]}$$

Since this is equivalent to (35), the outline of the proof of Theorem 10 is completed.

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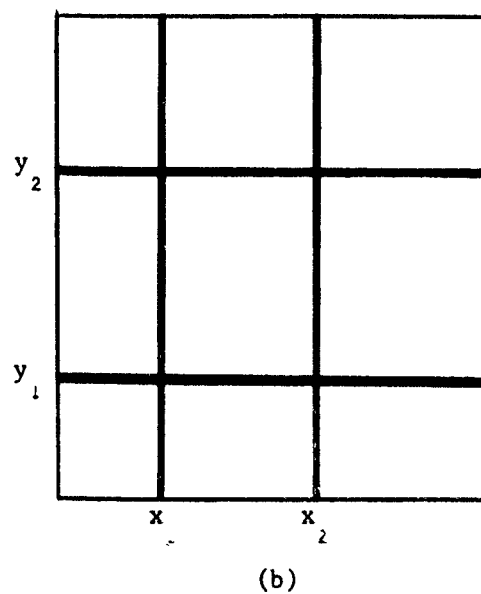
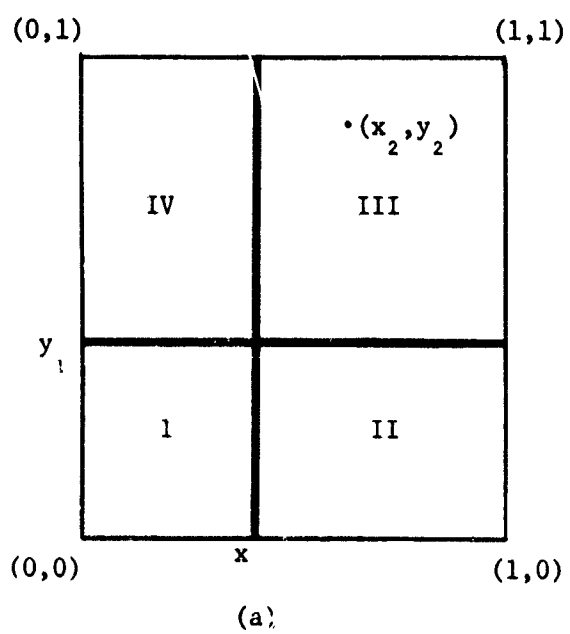


Figure 1

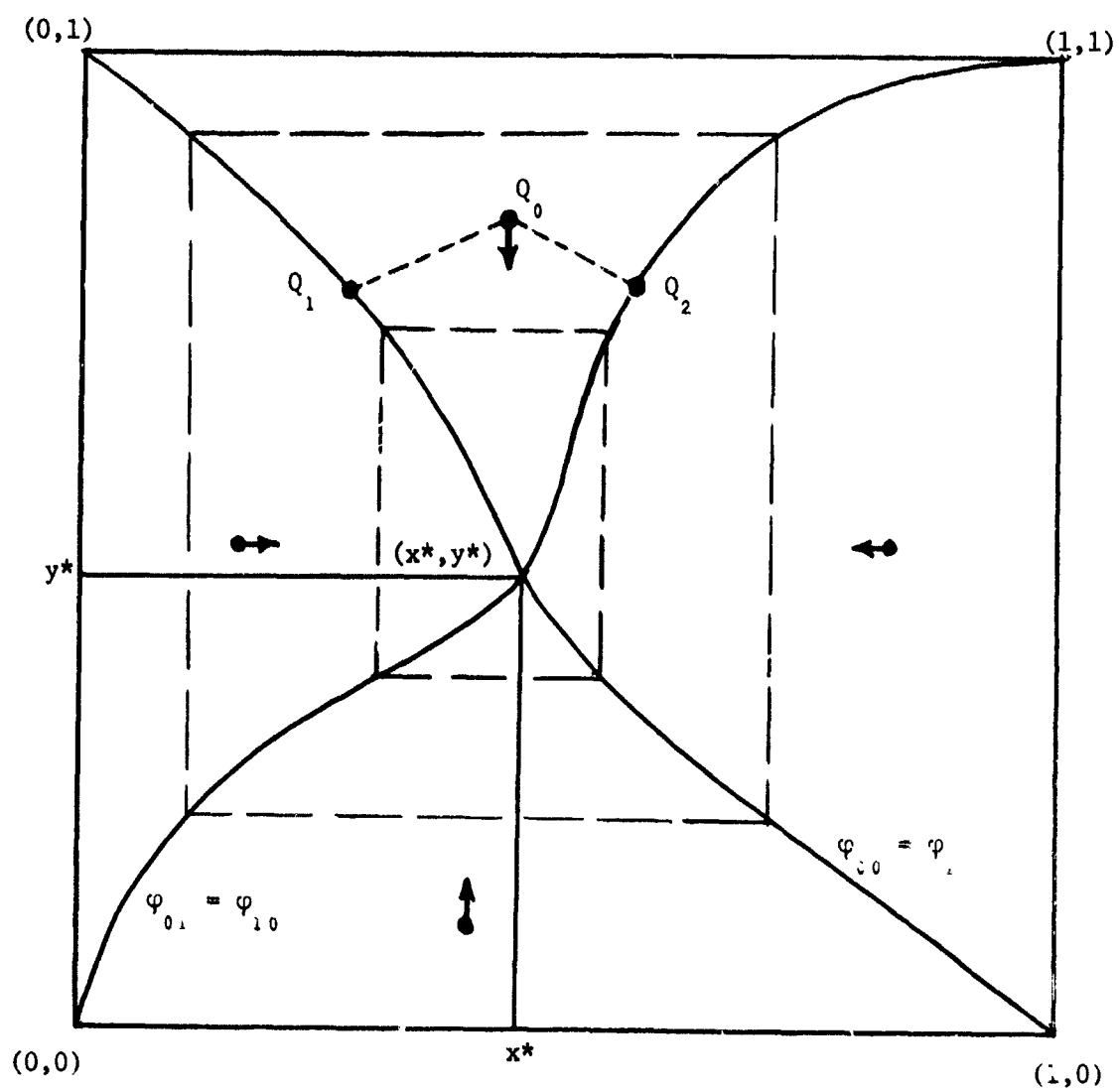


Figure 2. Aspects of Conservative u

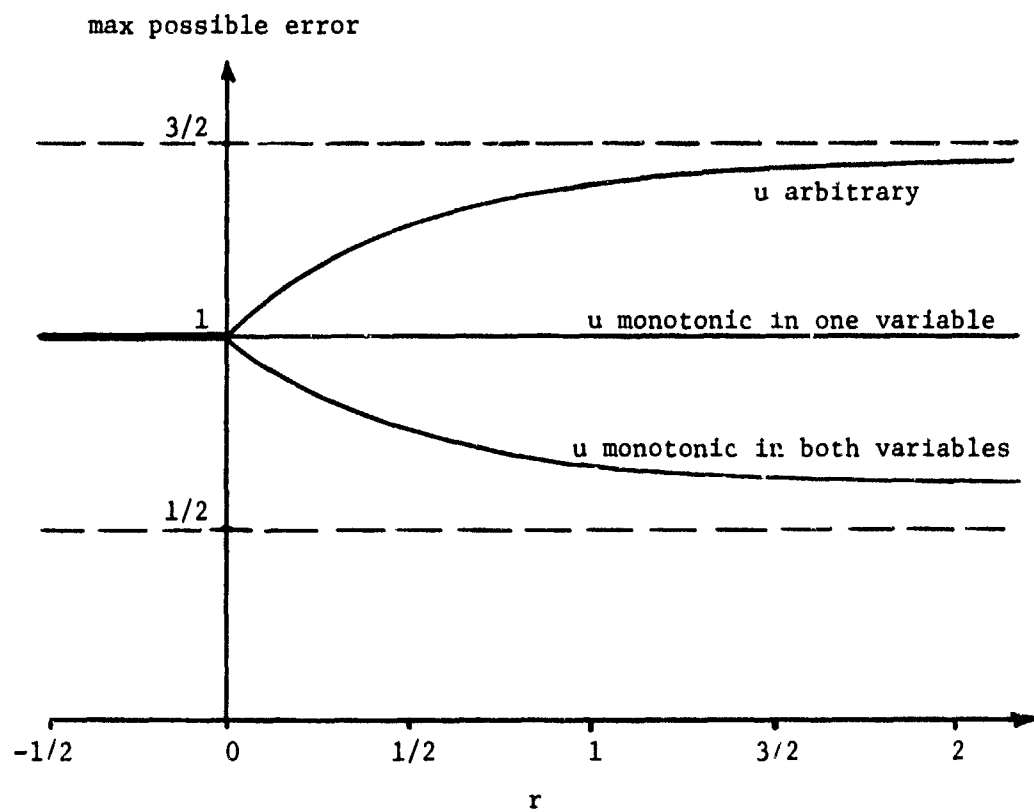
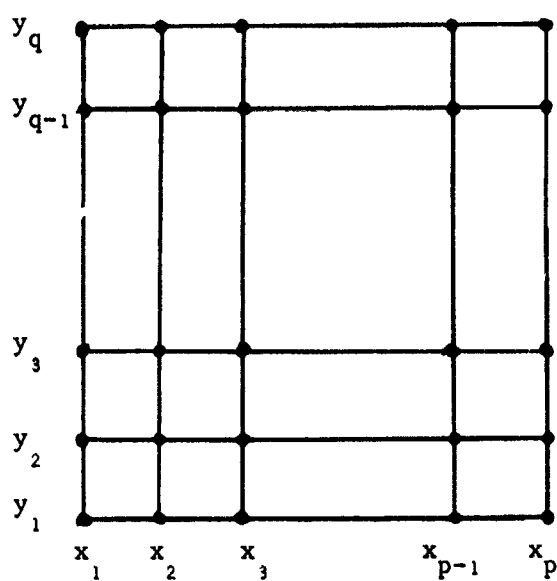
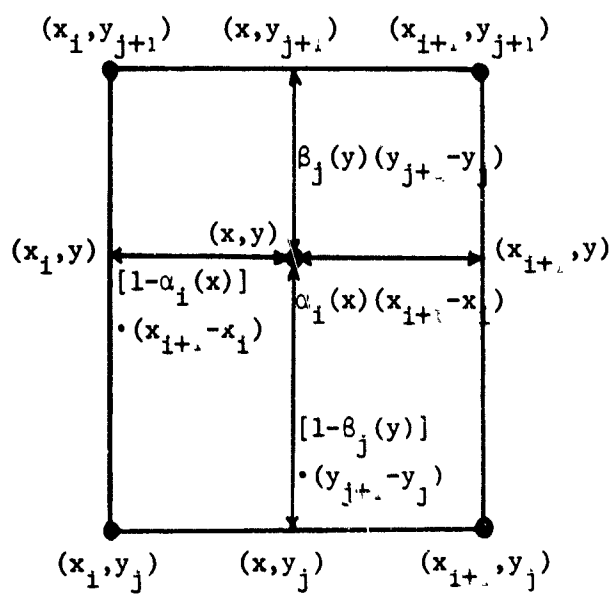


Figure 3. Upper bounds on $\|v - u\|$ when $\min u(x,y) = r$ and $\max u(x,y) = r + 1$.



(a)



(b)

Figure 4

1 Oct 72

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